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THE

DOCTRINE

OF

LIFE ANNUITIES

AND

ASSURANCES

THE
DOCTRINE
OF
LIFE-ANNUITIES
AND
ASSURANCES.

Printed by Richard Taylor and Co., Shoe Lane, London.

THE DOCTRINE OF LIFE-ANNUITIES AND ASSURANCES,

ANALYTICALLY INVESTIGATED AND EXPLAINED.

TOGETHER WITH SEVERAL USEFUL

TABLES

CONNECTED WITH THE SUBJECT:

AND A VARIETY OF

PRACTICAL RULES

FOR THE ILLUSTRATION OF THE SAME.

By FRANCIS BAILY,

OF THE STOCK-EXCHANGE.

LONDON:
SOLD BY JOHN RICHARDSON, ROYAL-EXCHANGE.

1810.

PREFACE.

IN the year 1808, I published a treatise on the *Doctrine of Interest and Annuities*, wherein I entered into a full investigation of all the principles relative to that science; together with its application in the various questions arising from any commercial, political, or financial inquiries. In the preface to that work I signified my intention of prosecuting the subject still further, so as to take in the whole Doctrine of *Life-Annuities and Assurances:* the present treatise, therefore, must be considered as a continuation of the work above alluded to; and will, I believe, contain all that is useful or interesting on the science. The motives, which induced me to submit the *former* work to the public, were there fully explained; and will equally, if not more forcibly, apply to the *present* treatise.

The importance of the subject, at the present day, cannot be doubted; since the greater part of the property of this kingdom is, in one shape or another, connected with this science. The present possessors of *entailed* estates are, in the common law, justly called *tenants for life*; and the same appellation may be given to those who hold by *curtesy* or by *dower*; marriage settlements also, and wills, generally determine the possession and reversion of estates to *particular lives:* and to these contingencies every freehold estate in the kingdom is liable. If to these we add the immense number of *copyhold* estates determinable on lives, and the estates possessed by *ecclesiastical* persons of every description (all of which will probably be ever subject to the same tenure), we shall find that the value of the greater part of the *real* estates in this country will be determinable upon the principles laid down in the present work.

The incomes, likewise, annexed to all places, civil and military; all pensions, and most charitable donations;—these, and others of a like kind, are annuities for *life*. Moreover, the dividends arising from a great part of the capital in the Public Funds are, by the wills of the donors, and from other causes, rendered of the same nature. Besides which, many *life-annuities* have been granted by Individuals, by Parishes, by Corporate Bodies, and by the Government itself. So that a great part of the *personal* estate, also, of this country is involved in a consideration of this subject.

In addition, however, to the cases above alluded to, there are various other circumstances in which this science will be found highly interesting and useful. There are many parents, at the present day, who are desirous of providing *Endowments* for their children, against they arrive at particular periods of life, when a sum of money is most frequently wanted: such as the time of their apprenticeship, or when they come of age, &c. Several of the Assurance Offices, lately established in London, have published the rates at which they will guarantee such sums: and the present work will enable the public to determine how far it may be *prudent* to accept them. See page 361.

Another interesting part of this subject is connected with the various establishments in this country, under the two general divisions of Societies for the benefit of *Old Age*, and Societies for the benefit of *Widows*. These establishments, when founded and conducted on a true and proper basis, ought always to be encouraged: and can only be objected to when the management of their concerns is likely to fall into the hands of ignorant or designing men, who may be induced to sacrifice the permanent interest of the society to their own immediate benefit and advantage. The ruin of most of these societies may be attributed to their ignorance or neglect of the true mathematical principles upon which they ought to proceed; and, without an attention to which, no establishment of this kind can possibly flourish.

But, the most important branch of this science is that of *Assurances:* which is still more extensive than either of those above mentioned. For, independent of the different classes of persons holding property under the several tenures alluded to in the beginning of this preface, and whose incomes will consequently determine with their lives, there is an immense number of other persons, in the different departments of society, subject to the same contingency. Every man engaged in either of the three *professions*, whose emoluments arise from his own personal abilities and exertions,—every one pursuing a *naval* or *military* life, whose income will cease at his death,—every person engaged in *manufactures*, *commerce*, or any other employment, whose own immediate exertions are the support of the concern in which he is engaged,—these and many others, too numerous here to insist on, will often be desirous of sacrificing some part of their present emoluments and profits, not only with a view to secure a suitable provision for their families at their decease, but likewise to render their own lives more easy and comfortable:

under the pleasing consolation that they have guarded against one of the great evils of a premature death.

Independent, however, of this general view of the subject, there are various other purposes for which Assurances are effected. Persons holding *Leases on lives*, and paying a *fine* on renewal, are oftentimes induced to insure a sum of money upon those lives, in order that they may be enabled to pay such fine when it becomes due. Some consider it a good method of securing a dubious or protracted *debt*, by assuring the life of the debtor. Others, again, may be entitled to an estate, or to a sum of money, at the end of a given term, or on the happening of a particular event, provided they be then alive to receive it: and, in order to secure such sum to their families, may be desirous of insuring their lives for such *term*, or against such *contingency*.

These, and a thousand other cases of daily occurrence, render this branch of the science interesting to every class of the community. Numerous Offices have lately sprung up, in the metropolis, for the purpose of granting Assurances on every possible contingency amongst lives in general; and it therefore becomes every one, engaged in the public business of life, to study this subject with attention.

But, notwithstanding the importance and utility of these inquiries, it is not much more than a century that they have been conducted in a proper and scientific manner*. The celebrated Dr. Halley led the way in England; and in his paper, inserted in the *Philosophical Transactions* for 1693, pointed out the true method of calculating the value of Annuities on *Lives*. In the pursuit of this object, he assumed the rate of human mortality for five successive years, as observed at *Breslaw*; and, from these data, formed the *first* correct table of the value of Life-Annuities. That table, however, being adapted only to every *fifth* year of human life, and calculated at only *one* rate of interest, was consequently very limited in its application and utility.

The illustrious De Moivre improved on what Dr. Halley had begun†. He carefully examined the table of observations given by that celebrated philosopher: and, finding that for several years together the decrements of life were uniform, and that it was only in youth and in old age that any considerable deviation occurred, he founded his ingenious hypothesis that the decrements of life are *equal* and uniform, from birth to the utmost extremity of human life. He was, at first, inclined to compose a Table of the values of Life-Annuities, by keeping close to the table of observations; that is, by dividing the whole extent of human life into *several* intervals, according to the difference of the decrements during those periods. But, before he undertook this task he tried what would be the result of supposing those decrements *uniform* from the age of twelve to the utmost extremity of life: and was satisfied that the excesses, arising on one side, would

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* Soon after the Revolution in this country many of the *loans*, for the service of Government, were raised upon Life-Annuities: and nothing can show more forcibly the low state of the science, at that period, than the vague manner in which the values of such annuities were estimated.

† Abraham De Moivre was born at Vitri, in Champagne, in 1667. The revocation of the Edict of Nantz, in 1685, determined him, with many others, to take shelter in England; where he perfected his mathematical studies, the foundation of which he had laid in his own country, and which have rendered him so great an ornament to the age in which he lived. In the latter part of his life he subsisted chiefly by giving answers to questions in Chances,

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Annuities, &c: and it is said that most of these solutions were delivered at a coffee-house in St. Martin's-Lane, where he spent the greatest part of his time. His merit and abilities were so well known and esteemed, that the Royal Society of London judged him a fit person to decide the famous contest between NEWTON and LEIBNITZ, concerning the invention of Fluxions. He was highly esteemed by the first of these celebrated philosophers: and it is reported that, during the last ten or twelve years of NEWTON's life, when any person came to ask him for an explanation of any part of his works, he used to say "Go to M. De Moivre; he knows all these things better than I do!" He died at the advanced age of eighty-seven.

be compensated by the defects on the other. For, on comparing his calculation with that of Dr. Halley, he found the conclusions to differ so very little, that he thought it superfluous to join together several different rules in order to compose a single one.

The first edition of his *Annuities on Lives* was printed in octavo, in 1724. By the most simple and elegant formulæ he pointed out the method of solving all the most common questions relative to the value of Annuities on single and joint lives, Reversions and Survivorships. In the subsequent editions of that work* he not only corrected the errors into which he had fallen in the first edition, but also greatly enlarged the boundaries of the science, and encouraged other mathematicians to pursue the path which he had struck out with so much honour to himself. Unfortunately, however, his hypothesis will not suit all circumstances: and more recent discoveries, on the rate of human mortality, have proved that it cannot always be safely adopted. Nevertheless it is still of great use in the investigation of many cases connected with this subject; and will ever remain a proof of his superior genius and abilities.*

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* The *second* edition appeared in 1743; and the *third* in 1750. Since which time, I believe there have been other editions: but the most improved copy is that which is inserted at the end of his *Doctrine of Chances*, 3rd edit. 1756.

In the preface to the second edition, here alluded to, he made an illiberal and unjustifiable attack on Mr. Simpson; and charged him with mutilating his propositions, obscuring his demonstrations, and pirating his rules. But Mr. Simpson *effectually* refuted these charges (in the same year) in an *Appendix* to his Doctrine of Annuities: at the close of which he exclaims, in the language of conscious rectitude, "I appeal to all mankind, whether, in his "treatment of me, he has not discovered an air of self-sufficiency, "ill-nature, and inveteracy, unbecoming a gentleman." Here the controversy appears to have dropped. For, M. De Moivre published the third edition of his book, without any further notice of Mr. Simpson: but omitted the offensive reflections which had been inserted in the preface to the preceding edition.

In the year 1742, Mr. Thomas Simpson published the first edition of his little treatise on the *Doctrine of Annuities and Reversions;* in which he introduced the method of computing such values from the *real observations* of life. His rules upon this subject are general, and will apply to any observations: nevertheless he confined himself, in his Tables and in his Examples, to the rate of human mortality in *London* as deduced from the observations of Mr. Smart†. The same author prosecuted this subject, by way of *Sup-*

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* The Tables of the value of Life-Annuities, deduced from M. De Moivre's hypothesis, will be found at the end of the present work; being Tables XLIX and L.

† Mr. Simpson's Tables of the value of Life-Annuities will be found at the end of this work; being Tables XLV to XLVIII.

plement, in his *Select Exercises for young Proficients in the Mathematics*, published in the year 1752.* This work, however, is, for the most part, a repetition of the rules given in the preceding treatise: to which are added some new problems on the subject of *contingent* annuities and assurances.

On the style of Simpson (always simple and elegant) it is needless for me to make any observations. His works are his best comment; and, need only be read, to be admired. Nevertheless his Treatise on Life-Annuities, together with his Supplement above alluded to, are perhaps the most imperfect of his productions. Although much is there done, still much more remained to be executed. His Tables, being deduced from the rate of mortality in *London* only, are found not to be sufficiently adapted for general use: and his Rules, being deduced partly from the hypothesis of M. De Moivre and partly from real observations, have been ascertained not to be sufficiently correct. Subsequent improvements in the science have also shown that some of his general theorems are erroneous: and that many cases, which frequently occur in practice, are not even mentioned in either of his works.

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* That part relating to Annuities has lately been taken out of the *Select Exercises;* and, having been printed separately in 1791, is now generally bound up with his *Doctrine of Annuities and Reversions:* the *second* edition of which appeared in 1775, and which contains the *Appendix* alluded to in the note in page xii.

In 1753, Mr. James Dodson published the Second volume of his *Mathematical Repository*; in which are contained not only the algebraical solutions of the problems given by M. De Moivre in his treatise above mentioned, but also several new and useful Questions connected with this subject: the Third volume of the same work appeared two years afterwards (1755). In the compass of two small duodecimo volumes, the author has contrived to solve an immense variety of Questions relative to Annuities, Reversions, Survivorships, and Assurances. The methods which he has pursued in investigating these cases are, in general, a model of analytical reasoning; and afford an excellent *praxis* for the young mathematician. Nevertheless he is sometimes obscure, from the use of uncouth symbols; and from the culpable practice of changing the signification of his characters during the course of the same investigation. In all his solutions he adopted the hypothesis of his friend De Moivre; conceiving that it would lead to more accurate results than the use of the Table of Life-Annuities formed by Mr.

Simpson (from the bills of mortality in *London*), the only one, at that time, deduced from real observations. The reader therefore will look in vain for any *correct* solution in the works of this author: although they may be occasionally referred to, for the method of finding an *approximate* value.

The science remained in this state, without much improvement, till the publication of the first edition of Dr. Price's celebrated treatise in 1769. This work, entitled *Observations on Reversionary Payments, &c*, was first published with a view to oppose and destroy the injurious effects and evil intentions of a class of men (unfortunately to be found in every stage of society) who, under pretence of establishing Societies for the *benefit* of Old Age, and of Widows, were only forming schemes to allure, and to defeat the hopes of the ignorant and the distressed. His efforts were eventually crowned with success: and those *bubble* societies have long since met with the fate which he so truly predicted.

In this laudable pursuit, Dr. Price saw the necessity of more accurate observations on the mortality of human life; in order to determine with more correctness the value of Life-Annuities, and to show more forcibly the futility and extravagance of the schemes that were issued by those societies*. By the assistance of some public-spirited individuals he obtained correct registers of the rate of mortality at Northampton, Norwich, Chester, and other places in England. But, still, the computation of the values of annuities, according to these observations, was a work so tedious and unpleasant, that little hopes were entertained of profiting by those researches: and Dr. Price suffered three several editions of his treatise to pass over without affording any additional information on this subject. At length the fourth edition appeared (1783) enriched with several valuable tables of Annuities on Single and Joint lives, at different rates of interest, deduced not only from the probabilities of living as observed at *Northampton*, but also from the probabilities of living as observed in the kingdom of *Sweden* at large†.

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* Any person, who will take the trouble to go through the Examples inserted in Dr. Price's treatise, will readily observe how inaccurately he was obliged to proceed in this infant state of the science. In calculating the value of *deferred* annuities, (a case of frequent occurrence,) he was obliged to take the *value of the annuity* from M. De Moivre's tables, but, the *probabilities of life* he deduced from Dr. Halley's table of observations at Breslaw. A practice which gives an air of imperfection to the work at the present day; and which *ought* to have been removed, after the publication of the late valuable tables.

† These tables are inserted at the end of the present work;

The great addition which Dr. Price has made to our means of information respecting this science, and the assiduity with which he thus promoted some of the best interests of mankind, deserve the highest commendation: and his labours on this subject entitle him to our warmest praise. The primary object, which he had in view, has been fully answered; and his treatise was admirably adapted to *that* end. In every other respect, however, it is far from being complete: and the reader will look in vain for the most common cases that occur in practice. Indeed, those subjects, which are to be met with, do not readily present themselves; owing to the loose and irregular manner in which they are treated. Dr. Price's object was not so much to insert what was *new*, as to illustrate (by some striking examples) a few of the leading problems, with a view to oppose the pernicious schemes that disgraced the age in which he lived. But, those schemes having long since vanished, his observations may now be considered rather as a *beacon* to posterity*.

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being Tables XIV to XLIV. Their publication forms a new era in the science; and has, in a great measure, rendered obsolete and useless the preceding works of De Moivre, Dodson, Simpson and Morgan. This latter treatise had been published about four years before this period; as will be mentioned in the sequel.

* Any person, the least acquainted with the subject of the pre-

The next treatise on this subject is that by Mr. Morgan, entitled *The Doctrine of Annuities and Assurances*, which appeared in 1779. This author sets out with the vain attempt to render the principles of the science intelligible to persons *unacquainted* with mathematics: but, after a fruitless effort for this purpose, he ultimately leaves his readers to pursue their inquiries by the common and only useful method of *analysis*. Besides some valuable observations "on the "different methods of determining the state of a Society, whose business consists in making Assurances "on Lives," that work will be found to contain a variety of problems treated, for the most part, in a plain, easy and familiar manner; and adapted to the state of the science at *that* period. But, out of the forty-two problems which that treatise contains, about thirty of them, chiefly relating to *contingent* annuities and assurances, are (owing to more accurate observa-

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sent work, must be aware that any additional *Tables* of the value of Life-Annuities, or any *Observations on the best method of forming them*, will add greatly to our means of information. It will therefore readily be seen that my remarks do not allude to this part of his treatise, which I consider invaluable, and of constant utility. My observations, in the present instance, apply more particularly to any improvement in the *analysis* of the science; and its application to any practical cases.

tions and a more improved analysis) *now* rendered totally unfit for general use. Mr. Morgan himself, however, has been the principal cause of this revolution in the science: but of the *merit* of his improvements on this subject I shall speak hereafter*.

The last professed treatise on the science, which I think worthy of notice†, is Mr. Baron Maseres's *Prin-*

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* In Mr. Morgan's *Doctrine of Annuities &c*, we find three *new* tables of the value of Life-Annuities, deduced from the probabilities of life as observed at Northampton: namely, one for Single lives, another for two Joint lives whose ages are equal, and another for two Joint lives whose difference of age is sixty years: the interest in each table being at 4 per cent. In this infant state of the science, every additional table contributed greatly to the means of information on this subject.

It may be here necessary to remark that the fourth edition of Dr. Price's *Obs. on Rev. Pay.* (which first contained the present valuable collection of Tables) did not appear till four years *after* the publication of Mr. Morgan's work above alluded to. So that, till within these thirty years, there existed only four tables of the value of Life-Annuities: viz. two founded on M. De Moivre's hypothesis, and two deduced from the *London* observations; which are the Tables XLVII to L at the end of this work. As to the table formed by Dr. Halley, it was too contracted for any real use.

† I must not however omit mentioning the name of Dr. Waring: who, in a small pamphlet entitled *On the Principles of translating Algebraic Quantities into Probable Relations, &c* (1792), has devoted about thirty pages to the consideration of the subject of Annuities and Assurances. His style and manner, however, will not be much admired by those who have read the works of preceding writers on this subject. The formula in page 61, is taken from that pamphlet.

ciples of the Doctrine of Life Annuities (1783): wherein this celebrated author has explained the subject in so familiar a manner, as to be intelligible even to those who are unacquainted with the Doctrine of Chances, and who have made no great proficiency in mathematics. This treatise, however, (although consisting of more than 700 quarto pages) goes no further, in the analysis of the subject, than the first two problems in the present work: but its value is greatly enhanced by containing a variety of new Tables of the value of Annuities on Single lives, and on two Joint lives of different ages, deduced from the probabilities of living as observed by *M. De Parcieux* amongst the Government Annuitants in France; these being justly considered by the learned author as the most proper data whereon to found the value of Life-Annuities*. There are, moreover, in that treatise several interesting observations on the best method of providing Annuities for *Old Age*, and on various subjects of finance and political economy; which render it particularly valuable to those who are desirous of information on these important questions, and will perpetuate the name and abilities of this truly public-spirited writer.

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* These tables (with the exception of those on Single lives, at the several rates of 2, $2\frac{1}{2}$, 7, 8, 9, and 10 per cent interest) are inserted at the end of the present work; being Tables V to XIII. The original tables, however, are carried to *six* places of decimals:—a dubious effort!

Soon after the publication of the *fourth* edition of Dr. Price's *Observations on Reversionary Payments*, (which contained the valuable collection of Tables of Life-Annuities, deduced from the observations made at Northampton and in Sweden) Mr. Morgan was enabled to detect the inaccuracy of those rules which, not only Mr. Simpson and others had given for determining the value of *contingent* annuities and assurances, but also which he himself had deduced from the same principles, in his treatise above mentioned: and he immediately set about to correct them. His labours, on this subject, are contained in the several *papers* inserted by him in the *Philosophical Transactions* for 1788, 1789, 1791, 1794 and 1800. In the first volume, here alluded to, he has considered those cases only in which *two* lives are concerned: in the next two volumes, his object was to deduce the value of contingent assurances in all those cases where *three* lives are concerned, and which admit of a *correct* answer: and in the last two volumes he proposed to determine the value of contingent annuities and assurances in all the *remaining* cases of *three* lives.

Whoever will take the pains to read over those papers with attention, must be struck with surprise and regret at the *strange* and *confused* manner which Mr. Morgan has pursued, in order to obtain the solution of the several problems under consideration. No one, at the present advanced state of the science (with so many models of simplicity and elegance before him), could expect to see any mathematical inquiries conducted in so loose, so obscure, and so extraordinary a manner. The investigations are tediously and unnecessarily prolix; crowded with useless repetitions, and a variety of unmeaning quantities. All which might, indeed, be excused, if the resulting *formulæ* had been at once simple and correct: instead of which, we find the grossest errors committed, not only as to their *form*, but as to their *accuracy*. They are, for the most part, *unnecessarily* long; abounding with *useless* quantities, (which render their numerical solution exceedingly intricate and difficult); and oftentimes *at variance* with the particulars mentioned in the investigation: which, together with the *erroneous* manner in which they are printed, renders them of little or no use to the public. Most of his problems are investigated in *two* different ways, and are solved by the means of *two* distinct formulæ: but, notwithstanding the similarity of these methods is studiously kept from the observation of the reader, and although these double formulæ are, in each problem, totally different *in appearance;* yet they will be found in *all* cases to be precisely the same *disguised under different symbols!* A curious and interesting branch of the science has been thus strangely distorted and enveloped in mystery; a depraved taste in mathematical reasoning has been introduced; and (what is, by far, of the greatest importance) many *false solutions* have probably resulted from too great a dependance on the general formulæ*.

Mr. Morgan and myself are the only persons that have ever yet attempted to give correct solutions in the several cases of *Contingent* Annuities and Assurances. These cases have been fully investigated in the fifth and eighth Chapters of the following treatise. But, in conducting those investigations, I could not avoid a frequent reference to the preceding labours of Mr. Morgan on this subject: not only with a view of censuring the culpable method which he has adopted in pursuing his inquiries; but also, in order to obviate any objection that might be made to *my* formulæ, *because* they do not correspond with *his*. It is needless, however, in this place, for me to add to the comments which I have already made in the two chapters above alluded to*.

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* The *Philosophical Transactions* not being within the reach of every person, Mr. Morgan has inserted his *formulæ,* for the solution of the several problems here alluded to, in the last edition of Dr. Price's *Obs. on Rev. Pay.* note (P). But, the errors of the *original* are multiplied in the *copy:* and Mr. Morgan, if he studies his own reputation as a mathematician, had better expunge them altogether in future, than suffer them to remain in their present *worse than useless* state.

The above are the principal English† authors that

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* See the *Observations* in pages 195, 214, and 241; also the several *Notes* in pages 186, 187, 210, 257, 259, 261, 263, 273, 279, 288 and 308: where the charges, above insisted on, are fully explained and demonstrated. See likewise the Notes in pages 114, 118, 126 and 137.

† With respect to the *foreign* writers on this science, their productions are more numerous than ours, but their inquiries are not so extensive. The subject of Life Annuities was treated by Van Hudden of Amsterdam; and likewise by the celebrated Jean de Witt, in his treatise entitled *De vardye van de lif renten &c* (1671). M. Struyck also inserted, in the Introduction to his Universal Geography (1740), some conjectures on the state of human mortality; and a long treatise on the method of calculating the value of Life-annuities. But M. Kerseboom carried his researches much further, in his treatise published in 1748; and afterwards in 1752.

Whilst these inquiries were pursuing in Holland, M. De Parcieux was occupied with the same subject in France. In his *Essai sur la Probabilité de la Durée de la Vie Humaine* (1746) he has endeavoured to establish the rate of mortality which exists amongst Life-Annuitants *only*; and has adopted it as a proper standard for determining the value of Life-Annuities. But besides this important point, he has discussed a number of other interesting subjects connected with this science: and his work will be read with much profit and advantage. In 1779 M. St. Cyran published his have written on the subject of Life-Annuities and Assurances. They are few in number; and the whole of their productions, taken collectively, by no means contain a complete view of the science. And, more-

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Calcul des Rentes Viageres, which contains many useful and valuable Tables. M. De Parcieux (the nephew of the preceding author of the same name) published also a treatise on this subject, entitled *Traité des Annuités* (1781). But the most useful work on this science is that published by M. Duvillard, under the title of *Recherches sur les Rentes &c* (1787).

The researches of M. Wargentin and M. Sussmilch are well known in this country, from the frequent mention of their labours by Dr. Price, in his *Obs. on Rev. Pay.* The immortal EULER has also condescended to illustrate the first principles of this science in a paper inserted by him in the *Histoire de l'Acad. Roy. de Berlin* for 1760: wherein a method is given (similar to that of Mr. Simpson) for determining the value of an annuity on a life one year younger, from the value of an annuity on a life one year older. The same author has likewise inserted, in his *Opuscula Analytica* (1785), the solution of a question relative to Reversionary Annuities.

But, notwithstanding the list of authors which is here adduced, it will be found that, as far as the *analysis* of the subject is concerned, the science remained nearly stationary under their hands. Their inquiries, in this respect, were confined principally to the method of deducing the value of annuities, on single and joint lives, from given tables of observations; that is, to such subjects as are detailed in the second Chapter of the present work. Those useful and interesting parts of the science which relate to the subject of Reversions, Survivorships, and Assurances, together with their several applications to the various purposes of life, do not enter into any of the foreign treatises which I have had an opportunity of seeing.

over, the late improvements have rendered them, in a great measure, either obsolete or useless: and have shown the necessity of a general revision of the subject. Under these circumstances I was induced to form a new treatise, which should comprehend not only all that is useful and important in either of the preceding works, but also such additional information as a more improved analysis and more recent discoveries in the science have been able to afford.

The following is the outline of my plan.

The First chapter contains a few elementary principles of the *Laws of Chance;* some remarks on the *Probabilities of Life*, with an account of the several *Tables of Observations* made at different parts of the world; and an explanation of the general method adopted to express those probabilities in all cases. This preliminary chapter will prevent much unnecessary repetition in the course of the work.

The Second chapter shows the method of determining the *Value of Annuities* on any *Single* or *Joint* lives; on the *Longest* of any number of lives; &c, &c. The second corollary to the first problem is of considerable importance in enabling us to deduce, in a very easy and expeditious manner, the value of annuities, on any single or joint lives, from real observations*. For, it should be particularly observed that, Tables of such values being once formed, the solutions to the subsequent problems become extremely easy; since the formulæ are expressed in terms denoting the value of such annuities.

The Third chapter contains the four necessary problems for the solution of all cases of absolute *Reversionary annuities:* and at the end of that chapter I have selected all the possible cases of two and three lives, in order that they may be more easily referred to. The formulæ there given will be found of considerable utility also in enabling us to determine the value of the *Fines* that ought to be paid for the *Renewal of Leases* held on two or three lives: as I have fully explained in the Examples given in page 397.

The Fourth chapter comprehends various cases of annuities depending on *Survivorships* between two and three lives. These cases might have been considerably augmented, but without any real benefit: since the most frequent ones are there inserted; and any other (which may arise) is easily solved by the same method of proceeding.

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* The *Specimen,* inserted in page 36, is not the same as that given by Dr. Price in his *Obs. on Rev. Pay.*, but an improved method deduced from the formula there alluded to.

The Fifth chapter relates to such cases of *Contingent Reversionary Annuities* as could not, for want of some previous information, be inserted in the two preceding chapters: and I believe that the method of solution, which I have there adopted, will come nearer to the correct value than any that has hitherto been published.

The Sixth chapter treats of *Assurances:* a subject of great importance and extensive utility at the present day. A full explanation of the doctrine is given in the two problems inserted in that chapter.

The Seventh chapter contains the method of determining the value of annuities on *Successive lives;* the value of *Fines* in Copyhold estates held on lives; the value of *Presentations*, *Advowsons*, and things of a like kind. It likewise enables us to determine the value of the *Fines* that ought to be paid for Renewing or Exchanging any lives held on a lease originally granted for three lives *and afterwards for a number of years certain:* a practice pursued by several corporations in this country*.

The Eighth chapter is devoted to an investigation of the value of *Contingent Assurances;* wherein I have considered every possible case in which not more than three lives are concerned. In this branch of the science I flatter myself that I have made considerable improvements. I have divested the subject of all extraneous matter; have not introduced more cases than were absolutely necessary; have exposed the singular formulæ given by Mr. Morgan (the only person who has preceded me in these inquiries); and have, for the most part, introduced more correct expressions for the value of the several cases there alluded to.

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* See some singular errors and absurdities, into which the Corporation of *Liverpool* had fallen upon this subject, pointed out in page 418.

The three remaining chapters complete the *analysis* of the science, and relate to such subjects as could not properly be introduced into either of the preceding ones. The Ninth is confined to an explanation of the celebrated *hypothesis* of M. De Moivre: wherein its great utility and convenience, in many obvious cases, is defended against the recent attacks of Dr. Price and Mr. Morgan. The Tenth treats of the method of determining the value of life-annuities payable *Half-yearly*, *Quarterly*, &c: also of the value of life-annuities *Secured by Land:* and of the value of Assurances of sums of money payable *immediately* on the extinction of any given lives. The Eleventh shows the method of finding in *Annual Payments* the value of any Assurance or of any Deferred Annuity: problems which will be found of very extensive use in practice.

The Twelfth chapter contains a variety of very useful *Questions* connected with this subject: to which are added the *Rules* for the solution of the same; and a numerous collection of *Examples*. These are thrown together into one chapter for two obvious reasons: in the first place, by being separated from the body of the work, they do not interrupt the analytical investigations; and secondly, they may be used (together with the Tables which follow) by such persons as are not acquainted with mathematics. Consequently, the present work will be accommodated to the use of both classes of readers; and (although some repetition is unavoidably occasioned thereby) may be thus rendered doubly valuable. The questions in this chapter are such as most frequently occur: but, others of less public utility, or the solution to which could not be conveniently expressed in words at length, are to be met with in the body of the work, subjoined to the respective problems.

The Thirteenth chapter shows the direct application of the sixth, thirteenth and eighteenth questions, in the preceding chapter, to some of the most useful and important concerns of life: namely, to the method of forming the best *Schemes* for providing annuities for the benefit of *Old Age*, and for *Widows*. These observations are brought together under one head, in order that they might not interrupt the regular arrangement of the questions: and because it gives me, thereby, an opportunity of enlarging more fully on this very interesting subject.

The Last chapter contains an account of the various *Societies* or *Companies* that have been formed in this metropolis for the purpose of granting Annuities, Assurances and Endowments; together with a review of their several plans and constitutions. By this statement the world may be enabled to judge of their comparative merits, and will easily decide on their respective claims to public patronage. This additional chapter cannot be considered foreign to the nature of the present work. For, after explaining the principles of the science, and after entering at large into the value of such subjects, it becomes me not only to show *where* a person might apply who is desirous of negotiating for such things, but likewise to point out those Offices at which he may do it with the greatest *security* and *advantage* to himself. And I am the more induced to do this, as I have lately observed several mean attempts to mislead and deceive the public. Uninterested in, and unconnected with, any of these Societies, I have been anxious only to give a clear and unprejudiced account, deduced from their own plans and proposals, as submitted by them to the public*. At the end of this chapter I have also inserted a brief account of the recent plan, adopted by Government, for converting the 3 per cent stocks into Life-Annuities. The propriety and advantage of a measure of this kind, I had suggested in my *Doctrine of Interest and Annuities.*

Respecting the *Tables* of Observations, and of Life-

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* Whilst the present work was at the press, I struck off 500 copies of this Fourteenth Chapter, which I published as a pamphlet, in a separate form, under the title of *An Account of the several Life-Assurance Companies established in London.* The whole impression has been already sold: but I may probably reprint it at some future opportunity.

From the tenor of that pamphlet many persons have imagined that I have given an unqualified approbation of the plan and constitution of the *Equitable Society:* whereas my intention was merely to represent it as the *most* equitable and advantageous society *now existing.* The exclusion (except in very rare instances) of persons, assured *for a term of years,* from the benefits of the Society, as well as the use of disproportionate rates for a *particular class* of the assured, are not consonant to the original views and intentions of the Society. Moreover, the long interval between each division of profits, as well as the *mode* of dividing those profits, are not such as (in my opinion) ought to be adopted by any *new* society: although we might justly pause before any alteration could be safely proposed for the adoption of the Equitable Society in its present state.

Annuities deduced therefrom, which are inserted at the end of this work, I can only observe that they are a collection of all that have hitherto been published, in this country, on the subject: and that they are here carefully reprinted from the authorities mentioned in the note in page 41. The necessity of *other* tables I have often had occasion to mention; but I fear it will be long before I shall be able to congratulate the public on this point. It consequently becomes the more necessary to have all those, which *do* exist, brought into one view, in order that they may be more conveniently consulted. The Tables of the value of Assurances for Single and Joint Lives, according to the observations made by *M. De Parcieux* and in *Sweden*, have been calculated expressly for this work; and will be oftentimes found extremely useful.

It now remains only to explain and to justify the method of notation adopted throughout the present work: and (since it is, for the most part, novel in its nature) to obviate any objections that may be made to its introduction into the analytical science. It is needless for me, at the present day, to point out the great advantage attending a convenient and simple notation: since it is notorious to every one, conversant with mathematics, that some of the greatest improvements in the science (and in the most abstruse parts of it, too) have arisen from this very source. The expressions for the probabilities of life, inserted in § 23, are an improvement on the method adopted by Mr. Simpson. For, he denotes the *probabilities* of the life A continuing 1, 2, 3, &c, years, by the characters $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, $\overset{\prime\prime\prime}{a}$, &c: whereas I make $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, $\overset{\prime\prime\prime}{a}$, &c, to denote the number of *persons living* at the end of those years, whereby the probabilities of the life A continuing 1, 2, 3, &c years, will be represented by the *fractions* $\frac{\overset{\prime}{a}}{a}$, $\frac{\overset{\prime\prime}{a}}{a}$, $\frac{\overset{\prime\prime\prime}{a}}{a}$, &c. This method of denoting those quantities keeps the several parts distinct, which will be found of considerable use in many of the investigations which arise in the progress of the work. In the next place, I have selected the Greek characters α, β, γ, to denote the number of persons living *at an age* n *years older than* A, B, C, respectively; whereby the solutions of all cases relative to Deferred or Temporary annuities are rendered extremely simple, and easily to be remembered: all the investigations of such cases, by preceding authors, being rendered extremely confused and unscientific, for want of some general principle of this kind*. I cannot anticipate any objection to this measure.

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* See particularly the methods pursued by Dr. Price and Mr. Morgan in *Obs. on Rev. Pay.* notes (B), (C) and (D).

Throughout the whole of this work I have uniformly represented the lives, which are the subject of discussion, by the *Roman* capitals A, B, C, &c; reserving the *Italic* capitals *A*, *B*, *C*, &c, to denote the *value of an annuity* on those lives. This improvement, though apparently trifling, is however more correct than the uniform practice, hitherto pursued, of making the same letters denote two different quantities in the same investigation. I have also adopted a more accurate mode of expression, when speaking of the value of annuities on lives, than has hitherto been followed by preceding authors. I never use the ambiguous phrase *the value of a life*, when I mean to express the *value of an annuity* on such life: neither do I, when speaking of more than one life, leave it doubtful whether the expression alludes to the *joint* lives, or to the *longest* of such lives. These vague and equivocal expressions ought to be banished from all true science; and, they would scarcely have been considered worthy of refutation, had they not been so generally adopted by *every* author who has hitherto written on this subject (not excepting even the accurate Simpson himself): whereby a vicious and corrupt mode of expression has been introduced, without any corresponding advantage to compensate for so gross an error.

Since every series, which expresses the value of an annuity on the whole continuance of life, may be divided into two parts, which (separately taken) denote the value of a Deferred and Temporary annuity, and since those cases are of frequent occurrence, it becomes exceedingly desirable to be enabled to express such partial series by some general character. The method which I have adopted has been to enclose, in a parenthesis, the value of an annuity on the given lives; and to affix the Roman letter d or t (according as the annuity is Deferred or Temporary), in order to show that the series, represented by the quantity within such parenthesis, is not complete. This plan I have found, in a variety of cases, extremely useful and convenient. See pages 43 and 45.

In deducing the value of Deferred annuities, however, in any practical questions, it is always requisite to know the value of similar annuities on the same number of lives *older by the given term* than such lives: it is therefore useful to be enabled also to denote the value of such annuities by some convenient expression; as well in order to avoid circumlocution as to render the resulting formulæ more simple and elegant. Such values I have, throughout the following work, denoted by affixing the small Italic letter ^{q} to the characters denoting the value of annuities on the given lives. Whereby these new values may be easily distinguished from the values of annuities on the original lives; and their meaning may be easily retained in the memory. See page 43.

With respect to the new characters introduced in the twenty-seventh and twenty-ninth Problems, (see pages 186, 189, 190, and 211) they are adopted more for the sake of a convenient reference than for any analytical purpose. With this view, I have, in the use of them, chosen such as will most readily assist the memory: the life on which the assurance is made being always expressed by the largest letter; the smaller ones being added merely to denote the number and value of the lives on which the contingency depends. Thus, the value of an assurance on the life A, provided he dies the first of *two* lives A B, is denoted by $\mathcal{A}^{B}$: but, if it depends on his dying the first of *three* lives A B C, it is denoted by $\mathcal{A}^{BC}$. In like manner, the value of an assurance on the life B, provided he dies the first of *two* lives B C, is denoted by $\mathcal{B}^{C}$: but, if it depends on his dying the first of *three* lives A B C, it is denoted by $\mathcal{B}^{AC}$. For a similar reason also I have, in the thirty-fifth Problem, (see page 240) denoted the value of an assurance on the decease of A *or* B, provided *either* of them be the first that dies of three lives A B C, by the character $\mathcal{AB}^{C}$. It will be found that many of the problems in the eighth chapter may be expressed in a very simple manner by means of these symbols; which can be my only apology for introducing them. Mr. Morgan (who is the only person, besides myself, that has attempted to give correct solutions to such problems) has arbitrarily adopted, in *his* formulæ, the unmeaning letters E, Σ, M, Q, &c, to denote the values here alluded to. But, when compound quantities are represented by more simple expressions, those characters ought to be preferred which will, most readily and with the least effort of memory, bring to our recollection the original quantity intended to be expressed*:

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* This principle has guided me in the choice of the characters inserted in the Scholium to the first Lemma in the fifth chapter (see page 116). I mention these things merely to obviate the objections of some hyper-critics, who may probably complain of the work as being *disfigured by Greek characters, and uncouth symbols.*

——————— Si forte necesse est
Indiciis monstrare recentibus abdita rerum,
Fingere cinctutis non exaudita Cethegis
Continget; dabiturque licentia sumta pudenter.

Hor.

and I should have been much better satisfied if I could have pursued the same plan in the thirty-ninth and subsequent Problems.

Such is the nature of the present work, which will most probably terminate my labours on this subject. Much of my time is taken up in answering questions, which are laid before me for solution, relative to Annuities and Assurances. Those solutions are oftentimes different from such as arise from the ordinary rules and methods laid down by preceding writers: and it is on this account that I have been more particular in my inquiries on this subject; as well as desirous of explaining the cause of the difference, in order to remove any doubt as to their accuracy or propriety. The theorems, from which my practical rules are deduced, are strictly and mathematically demonstrated in the course of the present work: and in the *numerical* enunciation of those rules (when applied to the solution of such cases as are submitted to my consideration) I discard the *indiscriminate* use of the Life-Annuity Tables, deduced from the *Northampton* Observations; so generally adopted by the different Assurance Offices, and so much recommended by their immediate supporters.

The motives, which have influenced me to this determination, it is unnecessary here to enter into: since they are fully explained in the course of the present work. And I can only add that they will continue to be my rule of conduct, as long as I am appealed to, by the public, as an arbiter on these subjects.

FRANCIS BAILY.

Office, No. 13, *Angel-Court*,
Throgmorton-Street,
Feb. 12, 1810.

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THE
DOCTRINE
OF
LIFE ANNUITIES,
&c.

CHAPTER I.

On the Laws of Chance; and the Probability of Human Life.

§ 1. It is not my intention here to enter into a full investigation of the nature and laws of chance, but merely to explain those principles of the doctrine which are more essentially connected with the subject of the present work; in order to prevent any misunderstanding in the terms which are occasionally made use of.

§ 2. The *probability* of the happening of any event is to be understood as the ratio of the chances, by which that event may happen, to all the chances by which it may either happen or fail: and it may be expressed by a fraction whose numerator is the number of chances whereby the event may happen, and whose denominator is the number of chances whereby it may either happen or fail. Thus, if there be a chances for the happening of any event, and b chances for its not happening; then will the probability of such event taking place be truly represented by $\frac{a}{a+b}$.

§ 3. In like manner, the probability of any event failing (or of its not happening) may be expressed by a fraction whose numerator is the number of chances whereby it may fail, and whose denominator is, as before, the whole number of chances whereby it may either happen or fail. Thus, the probability of the above event failing will be truly expressed by $\frac{b}{a+b}$.

§ 4. Since the sum of the two fractions, representing the probabilities of the happening and of the failing of any event, is equal to unity; it follows that, one of them being given, the other may be found by subtraction. Thus, the probability of an event happening being denoted by $\frac{a}{a+b}$, the probability of the same event failing will be truly represented by $1 - \frac{a}{a+b} = \frac{b}{a+b}$; and *vice versâ.*

§ 5. If, upon the happening of an event, a person be entitled to a given sum of money, his *expectation* of receiving that sum has a determinate value before the happening of the event; and such value is ascertained by multiplying the present value of the sum expected by the fraction which represents the probability of obtaining it. Thus, if a person has a chances of obtaining, and b chances of losing a certain sum of money, the present value of which is equal to s, then will $s \times \frac{a}{a+b}$ denote his expectation of receiving such sum, and will be the true value of his interest therein*.

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* These principles may be more familiarly explained by the following example. Suppose that a person has 3 chances in 5 to obtain £100, the present value of his *expectation* is the product of £100 by the fraction $\frac{3}{5}$, and consequently it is worth £60. For, supposing that an event may equally happen to any one of 5 different persons, and that the person to whom it does happen should, in consequence of it, obtain the sum of £100, it is plain that the right which each of them in particular has upon the sum expected is $\frac{1}{5}$ of £100; which right is founded on this principle, that if the 5 persons concerned in the happening of the event should agree not to stand the chance of it, but to divide the sum expected among themselves, then each of them must have $\frac{1}{5}$ of £100 for his pretension. Now, whether they agree to divide that sum equally among themselves, or rather choose to stand the chance of the event, no one has thereby any advantage or disadvantage, since they are all upon an equal footing: and consequently each person's *expectation* is worth $\frac{1}{5}$ of £100. Let us further suppose that two of the five persons concerned in the happening of the event, should be willing to resign their chance to one of the other three; then the person to whom these two chances are thus resigned has now 3 chances that favour him, and consequently he has now a right triple of what he had before, and therefore his expectation will in such case be worth $\frac{3}{5}$ of £100. Now, if we consider that the fraction $\frac{3}{5}$ expresses the probability of obtaining the sum of £100, and that $\frac{3}{5}$ of 100 is the same as $\frac{3}{5} \times 100$, we must naturally fall into the conclusion laid down

§ 6. The probability of the happening of *several events*, that are independent of each other, is equal to the product of the probabilities of the happening of each event considered separately. Thus, if the probability of the happening of the first of any number of independent events be denoted by $\frac{a}{a+b}$; that of the second, by $\frac{c}{c+d}$; that of the third, by $\frac{e}{e+f}$; &c, &c: then will $\frac{a}{a+b} \times \frac{c}{c+d} \times \frac{e}{e+f} \times$ &c, denote the probability of the happening of all those events. And this expression, multiplied by the present value of the given sum, will denote the value of the *expectation* of receiving such sum on the happening of all those events*.

---

in the text, that the expectation of receiving any sum is determined by multiplying such sum by the probability of obtaining it: and though this method of reasoning is deduced from a particular case, it will easily be perceived that it is general, and applicable to any other case. See De Moivre's *Doctrine of Chances*, p. 3.

* For example: Suppose that, in order to obtain £100, two events must happen, the first whereof has 3 chances to happen and 2 to fail, and the second whereof has 4 chances to happen and 6 to fail: the value of the expectation will in such case be $\frac{3}{5} \times \frac{4}{10} \times 100 = 24$ pounds. The demonstration of which will be very easy if it be considered that, supposing the first event had happened, the expectation (then depending entirely upon the second) would, before the determination of the second, be worth $\frac{4}{10} \times 100 = 40$ pounds. We may therefore look upon the happening of the first as a condition of obtaining an expectation worth £40: but, the probability of the first event happening has been supposed $\frac{3}{5}$, wherefore the expectation sought for is to be estimated

§ 7. By a similar method of reasoning it will be evident that the probability of the failing of any number of independent events is equal to the product of the probability of the failing of each event considered separately. Thus, if the probability of the failing of the first of any number of independent events be denoted by $\frac{b}{a+b}$; that of the second, by $\frac{d}{c+d}$; that of the third, by $\frac{f}{e+f}$; &c, &c: then will $\frac{b}{a+b} \times \frac{d}{c+d} \times \frac{f}{e+f} \times$ &c, denote the probability of the failing of all those events. And this expression, multiplied by the present value of the given sum, will denote the value of the *expectation* of obtaining such sum on the failing of all those events.

§ 8. Moreover, the probability of the happening of *either* of any number of independent events is denoted by the difference between unity and the expression mentioned in the last article. For, since $\frac{b}{a+b} \times \frac{d}{c+d} \times \frac{f}{e+f} \times$ &c, denotes the probability that any given number of events shall fail, it follows (from § 4) that $1 - \frac{b}{a+b} \times \frac{d}{c+d} \times \frac{f}{e+f} \times$ &c, will denote the probability that they shall *not* all fail; but that

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by $\frac{3}{5} \times \frac{4}{10} = 100$; that is, by the product of the two probabilities of happening, multiplied by the sum expected. The same method of reasoning may be applied to the happening of three, or any other number of events, as may be seen more at large in the authors who have treated on this subject.

some one or other of them will happen. And this expression, multiplied by the present value of the given sum, will denote the value of the *expectation* of receiving such sum, on the happening of either of those events.

§ 9. In like manner, if the expectation of receiving any sum depends upon the happening of any number of independent events, and upon the failing of any number of other independent events, its value will be equal to the present value of such sum, multiplied by the probability of all the former happening, and also by the probability of all the latter failing. And from these principles, we may determine the value of an expectation depending on the happening or failing of as many independent events as may be assigned.

§ 10. Hitherto I have considered only such events as are independent of each other: but, if we wish to determine the probability of the happening of two events that are dependent on each other*, we must multiply the probability of the happening of one of them, by the probability which the other will have of happening when the first is considered as having happened: and the same rule will extend to the happening of as many events as may be assigned.

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* Two events are independent, when they have no connection with each other, and the happening of one neither forwards nor obstructs the happening of the other: as the continuance or failure of any given lives. On the other hand, two events may be considered as dependent, when the probability of either's happening is altered by the happening of the other; as the continuance or failure of the same life in different periods of its duration: see § 2.

§ 11. If there are several expectations upon several sums, it is evident that the expectation upon the whole will be equal to the sum of the expectations upon each. But if only one sum is to be received on the happening or failing of the given events, the method of determining the value of the expectation will be somewhat altered. The process, however, which is to be pursued in such cases will be more fully explained in the course of the present work: what has been already said being merely introductory to the various probabilities and contingencies that occur in the following sheets.

§ 12. Now with respect to the probability that a person of a given age will or will not live to any other given age, or till a certain sum of money granted him becomes due, it is obviously in all cases a matter of very great uncertainty, and will be often very different in different persons of the same age. The chance which a man of thirty years of age, who is in good health and lives a temperate and quiet life in the country, has to live twenty years, or till he is fifty years of age, is evidently much greater than that of another man of the same age, and of the same degree of health and vigour of body, who lives in a great city and in scenes of riot and dissipation; and it is likewise greater than that of another man of the same age and of the same degree of health and vigour, but who is going into an unhealthy climate to which he has not been accustomed: and still more evidently it is greater than that of another man of the same age who is of a weak and sickly constitution, or who by his daily occupation is exposed to many dangers of his life from which the generality of mankind is exempt; as is the case with soldiers and sailors in time of war or actual service. These are circumstances beyond the reach of calculation; and all that can be done by any general rules upon this subject is to estimate the degree of probability with which it may be reasonably expected that a person of any given age will live to any other given age, upon a supposition that he has neither a better nor a worse chance of so doing than the majority of other persons of the same age. This medium or average chance of living is determined by tables that exhibit the number of persons which, out of a certain number of children born (usually not less than a thousand), are found by a long series of observations to be living at the end of every subsequent year of human life to its extreme period: which period in some of the tables is carried to 86, and in others to more than 90 years. The instances of the prolongation of human life to 100 years, or more, are so few that they are not thought to be worth attending to in forming any general rules on this subject.

§ 13. Various observations on the mortality of human life have been made by different persons, and in different places; and several tables of the kind above-mentioned have been calculated and formed by the different writers on this subject, such as Dr. Halley, Mr. Thomas Simpson, M. Kersseboom, M. De Parcieux, Dr. Price, M. Susmilch, M. Wargentin, M. Muret, and others. But the same table of the probabilities of life will not suit every place: for, long experience has shown that all places are not equally healthy; or that the number of persons who die annually is different in different places. Dr. Halley formed his table from observations on the births and burials of the inhabitants of the city of *Breslaw* (the capital of the duchy of Silesia in Germany), during a series of five years; viz. from 1687 to 1691. Mr. Thomas Simpson, from observations on the bills of mortality in *London* for ten years, from 1728 to 1737*. M. Kersseboom, from the registers of certain assignable annuities for lives in *Holland*, which had been kept there for one hundred and twenty-five years, and in which the ages of the several people dying in that period had been truly entered. M. De Parcieux, from a similar use of the lists of the tontines in *France*, the numbers of which were verified by the *Nécrologes*, or mortuary registers, of several religious houses of both sexes*. Dr. Price, from a register of mortality kept at *Northampton*, for forty-six years, from 1735 to 1780: the same author has also formed a table from a similar register kept at *Norwich* for thirty years, from 1740 to 1769: another from a similar register kept by Mr. Gorsuch at *Holy Cross* near Shrewbury, for thirty years from 1751 to 1780: another from a similar register kept by Dr. Aikin at *Warrington* in Lancashire, for nine years, from 1773 to 1781: another from a similar list kept by Dr. Haygarth at *Chester*, for ten years, from 1772 to 1781: another from the register of mortality at *Vienna*, for eight years; another from the register of mortality at *Berlin*, for four years, from 1752 to 1755: another from a similar register at *Brandenburgh*, for fifty years, from 1710 to 1759: each of the last three being from tables given by M. Susmilch: also another from the tables of mortality at *Stockholm*, for nine years, from 1755 to 1763, as given by M. Wargentin: and another from seven different enumerations of the whole population of the kingdom of *Sweden*, each repeated at the end of three years, viz. in 1757, 1760, 1763, 1766, 1769, 1772, and 1775. M. Muret formed his table from registers kept in forty-three parishes in the district of *Vaud* in Switzerland, for ten years, from 1756 to 1765.

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* In Table I, at the end of the present work, the column denoting the probabilities of life in *London*, is not the one here mentioned; but a more correct one formed by Dr. Price, from the bills of mortality for ten years, from 1759 to 1768. The same author likewise formed one from the bills of mortality for ten years, from 1771 to 1780.

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* In the table alluded to in the last note, the column denoting the probabilities of life in *France*, is not the one given by M. De Moivre and Mr. Ferguson; but a more correct one published by M. Florencourt at the end of his Treatise on Political Arithmetick. Referring however to the last note, it may be necessary here to remark that in Tables III and XLV I have preserved the original values as given by M. De Parcieux and Mr. Simpson; because, the Life Annuity Tables being deduced therefrom, I consider them as the most proper to be used in conjunction therewith.

§ 14. All these tables differ from each other; and in many cases so materially as to leave us in great doubt whether the subject has attained that degree of accuracy and correctness to which it is capable of being carried. It should be observed that there are two sorts of data for forming tables of the probability of the duration of human life: one is furnished by the registers, or *bills of mortality*, which show the numbers dying at all ages; the other, by the proportions of deaths at all ages to the numbers living at those ages, as discovered by surveys or *enumerations*. Those tables which are deduced from the former of these data are correct only when there is no considerable fluctuation among the inhabitants of a place, and when the births and burials are equal: for, when there are more removals *from*, than *to* a place, and the births exceed the burials (as is almost always the case in country parishes and villages) tables so formed give the probabilities of living too low: and when the contrary happens (as is generally the case in cities and large towns) they give the probabilities of living too high. But tables formed from the latter of these data are subject to no errors: they must be correct whatever the fluctuations are in a place, and how great soever the inequalities may be between the births and the burials.

§ 15. Most of the tables above-mentioned have been deduced from the former of these data; and in most of them due allowances have been made, as far as circumstances would admit, for the fluctuations arising from emigration &c. But I believe there are no observations extant which will enable us to form tables from the latter of these data, except those published by M. Wargentin* of the population of the kingdom of Sweden: and it is much to be regretted that similar observations are not made in other countries.

§ 16. It is a singular circumstance that not only do *females* live longer than *males*, but *married* women live longer than *single* women. All the tables of observations intimate this: but the fact has been more fully confirmed by the observations made by Dr. Aikin at *Warrington*, and by Dr. Haygarth at *Chester*; each of whom kept distinct registers of the rate of mortality amongst males and females. Similar registers also were kept at *Stockholm:* and, in the enumeration of the whole population of the kingdom of *Sweden*, this circumstance was particularly attended to. These latter observations therefore being formed on such unerring principles, furnish sufficient data for calculating distinct tables of the value of annuities on lives among males and females, taken separately or conjunctly: and which tables might be applied with good effect in determining the value of annuities or assurances where the lives of *widows* are concerned*.

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* In the *Memoirs of the Academy of Sciences at Stockholm*, in 1776.

§ 17. The tables of observations most used in this country at present, are those which were formed by Dr. Price from the bills of mortality at *Northampton*: but they derive their importance principally from those numerous tables of the value of annuities on single and joint lives, which are computed therefrom; and which afford great facility to the solution of the various cases connected with this subject†. In every other point of view it must appear extremely incorrect to take the rate of mortality in one particular town, as a criterion for that of the whole country. The observations ought to be made on the kingdom at large, in the same manner as in Sweden; more particularly as, in the real business of life, the calculations are general and uniform, and adapted to persons in every situation. But till the legislature thinks proper to adopt some efficient plan for furnishing these data‡, we must rest contented with the laudable exertions of public-spirited individuals; and avail ourselves of the best light which they afford us on this subject.

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* See Tables XIV to XXIV.

† See Tables XXV to XLIV.

‡ By virtue of an Act of Parliament which passed in 1800, a survey was made of the population of this kingdom in the following year: but, from the manner in which the survey was formed and conducted, it has afforded little or no information respecting the subject of the present work.

§ 18. With respect to the several tables of mortality above-mentioned, I do not think that any of them (with the exception of those given by M. Kerssebooom, and M. De Parcieux) afford the proper grounds for calculating the value of *annuities*. For, it is evident that no person, in an ill state of health, or who is conscious of any thing in his constitution that might tend to the shortening of life, would give that value for an annuity which the tables indicate: neither am I inclined to think that he would become a purchaser at all. The lives, therefore, of such persons as *do* become annuitants will consequently be *good* lives: or a certain part only of the general mass of mankind. The principles, upon which M. Kersseboom and M. De Parcieux have formed their tables, enable us to ascertain pretty accurately the rate of mortality among this class of people: and therefore form a proper basis for determining the value of *annuities*.

§ 19. The same observations may be applied, with nearly the same propriety, to the method of computing the value of *assurances**. For, it is well known that the assurer endeavours to guard as much as possible against a *bad* life: and the law of the land justly punishes any fraud in this respect. Nevertheless, as avarice or negligence may induce a relaxation of duty, the lives on which assurances are made are more liable to be *mixed* than those on which annuities are granted. In either case, however, we might often deduce a more correct value from knowing the situation of life, the residence, and mode of living of the parties concerned.

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* This fact is indubitable: for " during the last 33 years, from " January 1768 to January 1801, the number of assurances on " single lives [at the Equitable Society] has been 83,201; of " which number sixty thousand five hundred and ninety-seven " have been on the lives of persons under 50 years of age, among

§ 20. But in many instances, both of annuities and assurances, the ages and conditions of the lives are so involved that we must proceed upon general

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" whom the deaths have been *fewer* than those in the Northamp-" ton table *in the proportion of four to seven!*" (See Dr. Price's *Obs. on Rev. Pay.* vol. ii. p. 443.) No fact can more clearly show the inaccuracy of those tables for *general* use? and though it may be *prudent* for an *Insurance company* to adopt them, as well as to make use of the lowest rate of interest in calculating the values of annuities therefrom (whereby large profits are secured to the society); yet the *public,* who have no interest therein, and who occasionally seek for information on this head, should be cautious in using them, unless they appear to be applicable to the case in question. The grounds, on which the calculations are made, ought to be as correct as the present state of information will allow, in order that the public may be satisfied with the accuracy of the result. The contrary, however is the fact: and Dr. Price himself has at length acknowledged it; although in rather a surreptitious manner. He introduces M. Kersseboom's and M. De Parcieux's tables of observations in order " that nothing " on this subject may be wanting;" as if it were a work of supererogation, and not one of the most essential as well as one of the most valuable parts of his treatise.

principles, without reference to the particular situation of the parties: and therefore, were it on this ground only, it would be extremely desirable to ascertain the rate of mortality in the kingdom at large. It would enable us to determine how far the tables, now in use, might be depended upon; and furnish the basis for others more numerous and comprehensive.

§ 21. For the information of the reader I have inserted a comparative view of all the principal tables that have been given of the rate of mortality in different parts of the world; being Table I. at the end of this work. The first column shows the ages, and the other columns the number of persons living at those ages, out of 1000 born* at the different places mentioned at the head of each column: and these places are arranged according to their degree of mortality amongst them. London and other cities are therefore placed first; and the rest in their order, as nearly as possible, to the most healthy, which are the country provinces. This table will consequently serve to illustrate, in a striking manner, the great difference between the duration of life in large *cities* and in the *country:* for it will be seen that, in proportion as we recede from the former, the probability of life is greater, and the chance of arriving at old age is considerably increased. Thus, it appears that, out of a thousand persons born at Vienna, not half of them live to be two years of age; whereas at Norwich, that number will live to be eight years of age; and at Holy Cross, they live to be above twenty-seven years of age; whilst in the province of Vaud in Switzerland, they live to be forty-one years old. It will also fully confirm the observation, which has been made in § 18, respecting the probability of living amongst those persons who purchase annuities on their own lives: for it appears from the observations of M. De Parcieux that the chance of living amongst a set of *government annuitants* is in almost every period of their existence much *greater* than amongst an equal number of indifferent persons *living in the most healthy part of the globe:* and which consequently shows that the Northampton tables are a very inaccurate index of the rate of mortality amongst a set of persons who purchase annuities on their own lives.

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* The original tables commence with numbers differing from each other; but are here reduced to the same number at the beginning, viz. 1000: by which mean we are enabled, by inspection, to compare the numbers together at any age, and immediately perceive the relative degrees of mortality at the several places given. The reader will observe that I have given *other* tables of the probabilities of life for *France, Sweden, Northampton* and *London,* together with the decrements or number of persons dying annually; which, being on a more enlarged scale, may be used with greater accuracy in the solution of the several problems which occur in the present work. These Tables are III, XIV, XXV, and XLV.

§ 22. But, however inaccurate these tables of observations may be, or however inapplicable to existing circumstances, the subject of the present work is not at all affected thereby. For, since the principles here laid down, and the rules thence deduced, are all treated *generally*, without allusion to any particular table of observations, the reader may apply them to any of the tables above mentioned; or to any others which may be hereafter found to be more correct, or more suited to any given circumstances.

§ 23. In any table of observations, therefore, which expresses the number of persons living at every age of human life, let the number of the living at the age of A be denoted by a; and those answering to the next succeeding ages in the table, by $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, $\overset{\prime\prime\prime}{a}$, &c, respectively. Further; let the number of the living at the age of B be denoted by b; and those answering to the next succeeding ages in the table, by $\overset{\prime}{b}$, $\overset{\prime\prime}{b}$, $\overset{\prime\prime\prime}{b}$, &c, respectively. Again; let the number of the living at the age of C be denoted by c; and those answering to the next succeeding ages in the table, by $\overset{\prime}{c}$, $\overset{\prime\prime}{c}$, $\overset{\prime\prime\prime}{c}$, &c, respectively. In like manner, let the number of persons living at the end of n years from the age of A be denoted by α; and those answering to the next succeeding ages in the table, by $\overset{\prime}{\alpha}$, $\overset{\prime\prime}{\alpha}$, $\overset{\prime\prime}{\alpha}$, &c. Also let the number of persons living at the end of n years from the age of B be denoted by β; and those answering to the next succeeding ages in the table by $\overset{\prime}{\beta}$, $\overset{\prime\prime}{\beta}$, $\overset{\prime}{\beta}$, &c. Moreover, let the number of persons living at the end of n years from the age of C be denoted by γ; and those answering to the next succeeding ages in the table by γ', γ'', γ''', &c. And so on for any other lives. Then, considering that, out of a persons alive at the age of A, only a' of them will be alive at the end of the year, it is evident that the number of chances for the life of A continuing one year will be a'; and that the whole number of chances for its living or dying will be a: consequently the probability that A will live to the end of the first year will be denoted by $\frac{a'}{a}$. And by a similar method of reasoning it will be seen that the probability of his living to the end of the second year will be denoted by $\frac{a''}{a}$; and of his living to the end of the third year, by $\frac{a'''}{a}$; and so on: for, a'', a''', &c, will be respectively the number of chances for the life continuing 2, 3, &c, years, and a will still be the total number of chances for his living or dying in any year. In like manner, the probability of the life B continuing 1, 2, 3, &c, years, will be respectively represented by $\frac{b'}{b}$, $\frac{b''}{b}$, $\frac{b'''}{b}$, &c: and that of the life C, by $\frac{c'}{c}$, $\frac{c''}{c}$, $\frac{c'''}{c}$, &c, respectively: (See § 2). By a similar method of reasoning it will be found that the probability of the life A continuing n, $(n+1)$, $(n+2)$, $(n+3)$, &c, years will be respectively denoted by $\frac{\alpha}{a}$, $\frac{\alpha'}{a}$, $\frac{\alpha''}{a}$,

$\frac{\alpha'''}{a}$, &c: and that the probability of the life B continuing the same periods will be respectively denoted by $\frac{\beta}{b}$, $\frac{\beta'}{b}$, $\frac{\beta''}{b}$, $\frac{\beta'''}{b}$, &c: and that the probability of the life C continuing the same periods will be respectively denoted by $\frac{\gamma}{c}$, $\frac{\gamma'}{c}$, $\frac{\gamma''}{c}$, $\frac{\gamma'''}{c}$, &c.

§ 24. Moreover, the probability of any two lives, A and B, continuing in being together for 1, 2, 3, &c, years, will be respectively denoted by $\frac{a'b'}{ab}$, $\frac{a''b''}{ab}$, $\frac{a'''b'''}{ab}$, &c: and that of three lives A, B, C, by $\frac{a'b'c'}{abc}$, $\frac{a''b''c''}{abc}$, $\frac{a'''b'''c'''}{abc}$, respectively: (See § 6). Further, the probability of any two joint lives A and B continuing in being together for n, $(n+1)$, $(n+2)$, $(n+3)$, &c, years, will be respectively denoted by $\frac{\alpha\beta}{ab}$, $\frac{\alpha'\beta'}{ab}$, $\frac{\alpha''\beta''}{ab}$, $\frac{\alpha'''\beta'''}{ab}$, &c: and that of three lives A, B, C, for the same terms, by $\frac{\alpha\beta\gamma}{abc}$, $\frac{\alpha'\beta'\gamma'}{abc}$, $\frac{\alpha''\beta''\gamma''}{abc}$, $\frac{\alpha'''\beta'''\gamma'''}{abc}$, &c.*

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* I would here observe that, when several lives are involved in a question, A is generally supposed (unless otherwise expressed) to be the *youngest;* B older than A; C older than B; and so on. This rule, however, is dispensed with in the fifth and eighth chapters of the present work.

§ 25. This being premised, it is evident (from § 4) that the probability of A *dying* before the end of the first year will be denoted by $1 - \frac{\overset{\prime}{a}}{a}$. For, since $\frac{\overset{\prime}{a}}{a}$ denotes the probability of his living to the end of that period, if we subtract this value from unity it will give the probability of his not living so long. And by a similar method of reasoning it will be found that $\left(1 - \frac{\overset{\prime\prime}{a}}{a}\right)$, $\left(1 - \frac{\overset{\prime\prime\prime}{a}}{a}\right)$, &c, will denote the probability of the same life dying before the end of the second, third, &c, year, respectively. In like manner the probability of the life B or C failing in 1, 2, 3, &c, years, will be respectively represented by $\left(1 - \frac{\overset{\prime}{b}}{b}\right)$, $\left(1 - \frac{\overset{\prime\prime}{b}}{b}\right)$, $\left(1 - \frac{\overset{\prime\prime\prime}{b}}{b}\right)$, &c: or $\left(1 - \frac{\overset{\prime}{c}}{c}\right)$, $\left(1 - \frac{\overset{\prime\prime}{c}}{c}\right)$, $\left(1 - \frac{\overset{\prime\prime\prime}{c}}{c}\right)$, &c. Moreover, the probability that *either* of two lives A, B, or of three lives A, B, C, will fail in 1, 2, 3, &c, years, will be denoted by $\left(1 - \frac{\overset{\prime}{a}\overset{\prime}{b}}{ab}\right)$, $\left(1 - \frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{ab}\right)$, $\left(1 - \frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{ab}\right)$, &c; or $\left(1 - \frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}\right)$, $\left(1 - \frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}\right)$, $\left(1 - \frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}\right)$, &c, respectively. And universally, if we subtract from unity the probability of the lives continuing together to the end of the given term, the remainder will express the probability that they shall not all continue together to the end of that period; but that one or other of them will die previous thereto.

§ 26. But, the probability that *all* the lives A, B, C, &c, shall fail in one year will (by § 6) be denoted by $(1-\frac{\overset{\prime}{a}}{a}) \times (1-\frac{\overset{\prime}{b}}{b}) \times (1-\frac{\overset{\prime}{c}}{c}) \times$ &c; in two years, by $(1-\frac{\overset{\prime\prime}{a}}{a}) \times (1-\frac{\overset{\prime\prime}{b}}{b}) \times (1-\frac{\overset{\prime\prime}{c}}{c}) \times$ &c; in three years, by $(1-\frac{\overset{\prime\prime\prime}{a}}{a}) \times (1-\frac{\overset{\prime\prime\prime}{b}}{b}) \times (1-\frac{\overset{\prime\prime\prime}{c}}{c}) \times$ &c, &c. And the probability that this event shall not happen, but that some *one* or other of the lives shall continue in being to the end of the first, second, third, &c, year, will (by § 8) be represented by $1-(1-\frac{\overset{\prime}{a}}{a}) \times (1-\frac{\overset{\prime}{b}}{b}) \times (1-\frac{\overset{\prime}{c}}{c}) \times$ &c, $1-(1-\frac{\overset{\prime\prime}{a}}{a}) \times (1-\frac{\overset{\prime\prime}{b}}{b}) \times (1-\frac{\overset{\prime\prime}{c}}{c}) \times$ &c, $1-(1-\frac{\overset{\prime\prime\prime}{a}}{a}) \times (1-\frac{\overset{\prime\prime\prime}{b}}{b}) \times (1-\frac{\overset{\prime\prime\prime}{c}}{c}) \times$ &c, &c, respectively.

§ 27. Hitherto, in deducing the probability of a life failing in any given time, I have had regard only to such event taking place *at any time before* the end of that period: but if we wish to determine the probability of the life failing *in* any particular year the exegesis will be materially different. The probability that A will die in the second year, after having out-lived the first year, is evidently equal to $1 - \frac{\overset{\prime\prime}{a}}{\overset{\prime}{a}} = \frac{\overset{\prime}{a} - \overset{\prime\prime}{a}}{\overset{\prime}{a}}$; because $\frac{\overset{\prime\prime}{a}}{\overset{\prime}{a}}$ will *then* denote the probability of its living to the end of that year, and this value, being subtracted from unity, will give the probability of its *then* dying in that year: but since this event *now* depends upon its living through the preceding year (the probability of which is $\frac{\overset{\prime}{a}}{a}$) the value above found must be multiplied by such probability, in order to give its true value: whence, the *present* value of the probability that the life A will fail *in* the second year, is truly denoted by $\frac{\overset{\prime}{a} - \overset{\prime\prime}{a}}{\overset{\prime}{a}} \times \frac{\overset{\prime}{a}}{a} = \frac{\overset{\prime}{a} - \overset{\prime\prime}{a}}{a}$. In like manner the probability of its failing *in* the third year is expressed by $\frac{\overset{\prime\prime}{a} - \overset{\prime\prime\prime}{a}}{\overset{\prime\prime}{a}} \times \frac{\overset{\prime\prime}{a}}{a} = \frac{\overset{\prime\prime}{a} - \overset{\prime\prime\prime}{a}}{a}$: and so on to the n^{th} $(n+1)^{\text{st}}$, $(n+2)^{\text{nd}}$, $(n+3)^{\text{rd}}$, &c, years; when the probabilities of the given life A failing *in* those years will be respectively $\frac{\alpha - \alpha}{a}$, $\frac{\alpha - \overset{\prime}{\alpha}}{a}$, $\frac{\overset{\prime}{\alpha} - \overset{\prime\prime}{\alpha}}{a}$, $\frac{\overset{\prime\prime}{\alpha} - \overset{\prime\prime\prime}{\alpha}}{a}$, &c: and where $\underset{\prime}{\alpha}$ denotes the number of persons living at the end of $n-1$ years from the age of A. The same observations will apply to the case of any number of joint lives: for, by pursuing the same method of reasoning, it will be found that the present probability of *either* of the three lives A, B, C, failing *in* the second year will be denoted by $\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}-\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}\times\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}=\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}-\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}$; *in* the third year, by $\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}-\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}\times\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}=\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}-\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}$; and so on to the n^{th}, $(n+1)^{\text{st}}$, $(n+2)^{\text{nd}}$, $(n+3)^{\text{rd}}$, &c, year; when the probabilities of the lives failing *in* those years respectively will be $\frac{\underset{\prime}{\alpha}\underset{\prime}{\beta}\underset{\prime}{\gamma}-\alpha\beta\gamma}{abc}$, $\frac{\alpha\beta\gamma-\overset{\prime}{\alpha}\overset{\prime}{\beta}\overset{\prime}{\gamma}}{abc}$, $\frac{\overset{\prime}{\alpha}\overset{\prime}{\beta}\overset{\prime}{\gamma}-\overset{\prime\prime}{\alpha}\overset{\prime\prime}{\beta}\overset{\prime\prime}{\gamma}}{abc}$, $\frac{\overset{\prime\prime}{\alpha}\overset{\prime\prime}{\beta}\overset{\prime\prime}{\gamma}-\overset{\prime\prime\prime}{\alpha}\overset{\prime\prime\prime}{\beta}\overset{\prime\prime\prime}{\gamma}}{abc}$, &c: and where $\underset{\prime}{\alpha}\underset{\prime}{\beta}\underset{\prime}{\gamma}$ denote the number of persons living at the end of $n-1$ years from the ages of A, B, C, respectively.

CHAPTER II.

On Life Annuities in general.

§ 28. THE method of determining the present value of any annuity is, to find the present value of each year's rent as it becomes due; and the sum of all these will be the total present value of the annuity required. Such value will in all cases depend on the annual rate of interest concerned; and throughout the whole of the present work I have denoted this *annual* rate by ϱ*: consequently the *amount* of £1 at the end of a year will be denoted by $(1+\varrho)$; and the *present value* of £1 certain to be received at the end of 1, 2, 3, &c, years will be respectively denoted by $(1+\varrho)^{-1}$, $(1+\varrho)^{-2}$, $(1+\varrho)^{-3}$, &c.† The sum of which continued to n terms, or $(1+\varrho)^{-1} + (1+\varrho)^{-2} + (1+\varrho)^{-3} + \dots\dots (1+\varrho)^{-n} = \frac{1-(1+\varrho)^{-n}}{\varrho}$, will denote the *present value of an annuity* of £1 per annum for n years: and if this series be continued to infinity, the sum of it, or $\frac{1}{\varrho}$, will express the *present value of the perpetuity* of the same annuity. The principles, on which these observations are founded, have been fully explained in my treatise on the *Doctrine of Interest and Annuities*; but I have thought it necessary to mention them here, in order to prevent circumlocution in the investigation of the following problems.

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* The *annual* rate should, in all cases of compound interest, be carefully distinguished from the *nominal* rate: but, such annual rate may always be expressed in terms of the nominal rate, as I have distinctly shown in another work. See *Doctrine of Interest and Annuities*, p. 16.

† As we shall often have occasion in the course of this treatise to find the present value of £1 certain to be received at the end of any number of years, and also the present value of £1 *per annum* for any number of years, at different rates of interest; I have (in order to avoid the necessity of referring to any other work) inserted at the end, two tables, which show the values here alluded to: being Tables LVIII and LIX, at the end of this work.

§ 29. In life annuities, however, the rent of each year is to be received only on certain contingencies; consequently the present values above mentioned must be diminished in proportion to the probability of receiving them: and the sum of such values, for each successive year, will be the total present value of the life annuity required. Throughout the whole of this work I have supposed the annuity to be £1 per annum; in which case, the present value deduced will denote the *number of years purchase* that such annuity is worth: and which, being multiplied by any other annuity, will give the present value of such other annuity accordingly.

PROBLEM I.

§ 30*. To find the value† of an annuity granted upon any number of lives; that is, for as long as they shall all continue in being together.

SOLUTION.

Let A, B, C, &c, be the lives upon which the annuity is granted; and let the probability of each life continuing 1, 2, 3, &c, years, be as denoted in § 23: then it will follow, from what has been said in § 24, that the probability of all the lives continuing to the end of the first year will be $\frac{a'b'c'\,\&c}{a\,b\,c\,\&c}$: which, being multiplied by $(1+\varrho)^{-1}$, or the present value of £1 certain to be received at the end of one year, will produce $\frac{\overset{\prime}{a}\,\overset{\prime}{b}\,\overset{\prime}{c}\,\&c}{(1+\varrho)\,abc\,\&c}$ for the present value of the first year's rent; or the *expectation** of receiving such sum on the contingency that all the lives continue to the end of the first year. In like manner, since the probability that all the lives will continue to the end of the second year is $\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{b}\,\overset{\prime\prime}{c}\,\&c}{a\,b\,c\,\&c}$, if this expression be multiplied by $(1+\varrho)^{-2}$, or the present value of £1 certain to be received at the end of two years, it will produce $\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{b}\,\overset{\prime\prime}{c}\,\&c}{(1+\varrho)^2\,a\,b\,c\,\&c}$ for the present value of the second year's rent; or the *expectation* of receiving such sum on the contingency that all the lives will continue to the end of the second year. By the same method of reasoning it will be found that $\frac{\overset{\prime\prime\prime}{a}\,\overset{\prime\prime\prime}{b}\,\overset{\prime\prime\prime}{c}\,\&c}{(1+\varrho)^3\,a\,b\,c\,\&c}$ will denote the present value of the third year's rent; or the *expectation* of receiving such sum on the contingency that all the lives will continue to the end of the third year. And in this manner we must proceed for all the subsequent years of human life, the sum of all which terms†, or $\frac{1}{a\,b\,c\,\&c}\times$

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* This fundamental proposition, upon which the whole doctrine of annuities in a great measure depends, may be found in most authors who have treated on this subject. In the investigation of the subsequent Problems and their Corollaries, I shall refer to the similar propositions in the works of the five following authors: viz. Simpson's *Doctrine of Annuities and Reversions*, 1775; and his *Supplement* to the same, 1791. De Moivre's *Doctrine of Chances*, 3rd edit. 1756. Dodson's *Mathematical Repository*. Dr. Price's *Observations on Reversionary Payments*, 6th edit. 1803. Morgan's *Doctrine of Annuities and Assurances*, 1779: and his papers in the *Philosophical Transactions* for 1788, 1789, 1791, 1794, and 1800. Whereby the reader may the more readily compare them together, and judge of the respective merits of the rules which they have given for the solution of the same.

† By the term *value*, I mean the number of years purchase that the annuity is worth, agreeably to what I have just observed. This mode of expression will be used throughout the present work.

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* See Ch. 1. § 5 and 6.

† When the sum of this series is to be determined in numbers, the terms of it must be carried to the extinction of the *oldest* life C, involved in the combination; at which period all the subsequent terms vanish because $\overset{\prime}{\gamma}$, $\overset{\prime\prime}{\gamma}$, &c, become equal to nothing. This observation will apply to all cases of combined lives.

$$\left[\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}\ \&c}{(1+\varrho)}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}\ \&c}{(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}\ \&c}{(1+\varrho)^3}+\ldots\ldots\ \frac{\alpha\beta\gamma\ \&c}{(1+\varrho)^n}\right]$$ will be the total present value of the annuity: where n in this case denotes the number of years between the age of the oldest of the given lives, and the age of the oldest life in the table of observations.

COROLLARY I.

§ 31. Now, when only one life A is concerned, this series will become $\frac{1}{a}\left[\frac{\overset{\prime}{a}}{(1+\varrho)}+\frac{\overset{\prime\prime}{a}}{(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{a}}{(1+\varrho)^3}+\&c\right]$ when two lives, A and B are concerned, it will become $\frac{1}{ab}\left[\frac{\overset{\prime}{a}\overset{\prime}{b}}{(1+\varrho)}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{(1+\varrho)^3}+\&c\right]$. Hence, if we make the italic capitals *A*, *B*, *C*, &c, to denote *the value of an annuity* on any single life A, B, C, &c; and *AB, AC, BC*, &c, to denote *the value of an annuity* on any two joint lives AB, AC, BC, &c; and *ABC*, &c, to denote *the value of an annuity* on any three joint lives ABC, &c: then in the case of a single life, we shall have

$$\begin{array}{l}
\frac{1}{a}\left[\frac{\overset{\prime}{a}}{(1+\varrho)}+\frac{\overset{\prime\prime}{a}}{(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{a}}{(1+\varrho)^2}+\&c\right]=A\\
\frac{1}{b}\left[\frac{\overset{\prime}{b}}{(1+\varrho)}+\frac{\overset{\prime\prime}{b}}{(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{b}}{(1+\varrho)^3}+\&c\right]=B\\
\frac{1}{c}\left[\frac{\overset{\prime}{c}}{(1+\varrho)}+\frac{\overset{\prime\prime}{c}}{(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{c}}{(1+\varrho)^3}+\&c\right]=C\\
\quad\&c\qquad\quad \&c\qquad\quad \&c\qquad\qquad\qquad \&c:
\end{array}$$

and in the case of two joint lives, we shall have

$$\frac{1}{ab}\left[\frac{\overset{\prime}{a}\overset{\prime}{b}}{(1+\varrho)}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{(1+\varrho)^3}+\&c\right]=AB$$

$$\frac{1}{ac}\left[\frac{\overset{\prime}{a}\overset{\prime}{c}}{(1+\varrho)}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{c}}{(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{c}}{(1+\varrho)^3}+\&c\right]=AC$$

$$\frac{1}{bc}\left[\frac{\overset{\prime}{b}\overset{\prime}{c}}{(1+\varrho)}+\frac{\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{(1+\varrho)^3}+\&c\right]=BC$$

&c &c &c &c:

and in the case of three joint lives, we shall have

$$\frac{1}{abc}\left[\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{(1+\varrho)}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{(1+\varrho)^3}+\&c\right]=ABC$$

&c &c &c &c.

COROLLARY II.

§ 32. Although there is no method of summing up these terms, or of abridging the general expression above given for finding the value of annuities on single and joint lives, but each respective term must be actually reduced to numbers, and the whole of them added together in order to determine the total value of the annuity; yet in finding the value of annuities on a number of single or joint lives, that is on lives of several successive ages, the process may be considerably abridged by deducing the value of an annuity on the next younger life from the value of an annuity on a life or lives each one year older. For, let ABC denote the value of an annuity on any number of joint lives A, B, C*; and $\underset{\prime}{A}\underset{\prime}{B}\underset{\prime}{C}$ the value of an annuity on the same number of joint lives each one year younger than A, B, C; and let $\underset{\prime}{a}\,\underset{\prime}{b}\,\underset{\prime}{c}$ &c be the number of persons living at the ages of those younger lives, as found in any table of observations: then, for the very same reason that $\frac{1}{abc}\left[\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{(1+\varrho)} + \frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{(1+\varrho)^2} + \frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{(1+\varrho)^3} + \&c\right] = ABC$, we shall have $\frac{1}{\underset{\prime}{a}\underset{\prime}{b}\underset{\prime}{c}}\left[\frac{abc}{(1+\varrho)} + \frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{(1+\varrho)^2} + \frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{(1+\varrho)^3} + \&c\right] = \underset{\prime}{A}\underset{\prime}{B}\underset{\prime}{C}$. Wherefore, multiplying the first equation by abc, and the latter by $(1+\varrho)\,\underset{\prime}{a}\underset{\prime}{b}\underset{\prime}{c}$, we have

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* In the investigation I shall confine the case to *three* joint lives: but it is easy to see that the method is general, *whatever* number be taken.

$$\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{(1+\varrho)} + \frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{(1+\varrho)^2} + \&c = ABC \times abc$$

$$\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{(1+\varrho)} + \frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{(1+\varrho)^2} + \&c = \underset{\prime}{A}\underset{\prime}{B}\underset{\prime}{C} \times (1+\varrho)\,\underset{\prime}{a}\underset{\prime}{b}\underset{\prime}{c} - abc:$$

whence $\underset{\prime}{A}\underset{\prime}{B}\underset{\prime}{C}\,(1+\varrho)\,\underset{\prime}{a}\underset{\prime}{b}\underset{\prime}{c} - abc = ABC \times abc.$

Consequently $\underset{\prime}{A}\underset{\prime}{B}\underset{\prime}{C} = (1+ABC) \times \frac{abc}{\underset{\prime}{a}\underset{\prime}{b}\underset{\prime}{c}}(1+\varrho)^{-1}:$

whence the following rule for finding the value of an annuity on any *single* life, the principle of which it is easy to apply to the case of any *joint* lives. Begin with the oldest life in the table of observations; add unity to the value of an annuity on that life (usually equal to 0) and multiply the sum by the expectation of *a life one year younger* receiving £1 at the end of a year; the product will be the value of an annuity on the life one year younger: this value being substituted for the value of an annuity on the oldest life, and the process repeated, will give the value of an annuity on the next youngest life: and so on till we come to the age of the given life. Now, though the method of deducing the value of an annuity on any *given* life by means of this formula is rather more laborious than finding the numerical value of each term of the series given in the last corollary; yet the present formula has this advantage, that the several steps of the process give the values of annuities on lives of all the ages between the given life and the oldest life in the table of observations: whence, the calculation of the value of an annuity on lives of *all* those ages becomes scarcely more troublesome than the calculation of the value of an annuity on the youngest life.

§ 33. *Example* 1. Let it be required to find the value of an annuity on a life aged 90 years, allowing interest at the rate of 5 per cent per annum, and according to the probabilities of life in general as observed in *Sweden*.

The value of an annuity on a life aged 96 is evidently equal to 0:

$$\begin{array}{rl}
\text{aged } 95, = & (1+0) \times \frac{1}{2} \times {\cdot}9524 = {\cdot}4762 \\
94, = & (1+{\cdot}4762) \times \frac{2}{5} \times {\cdot}9524 = {\cdot}5624 \\
93, = & (1+{\cdot}5624) \times \frac{5}{11} \times {\cdot}9524 = {\cdot}6764 \\
92, = & (1+{\cdot}6764) \times \frac{11}{21} \times {\cdot}9524 = {\cdot}8363 \\
91, = & (1+{\cdot}8363) \times \frac{21}{33} \times {\cdot}9524 = 1{\cdot}1129 \\
90, = & (1+1{\cdot}1129) \times \frac{33}{47} \times {\cdot}9524 = 1{\cdot}4129
\end{array}$$

whence it appears that the value required is equal to 1·4129*.

§ 34. *Example* 2. What is the value of an annuity on the joint lives of a man aged 90, and a woman aged 84; reckoning interest at 4 per cent and the probabilities of living, among males and females *respectively*, as observed in *Sweden?*

By beginning with the oldest life in the table of observations, joined to another whose difference of age is 6 years, it will be found that the value of an annuity on two similar joint lives aged 95 and 89 is equal to 0:

$$\begin{aligned}
94 \text{ and } 88 &= (1+0) \times \frac{1 \times 76}{4 \times 99} \times \cdot 9615 = \cdot 1845 \\
93 \text{ and } 87 &= (1+\cdot 1845) \times \frac{4 \times 99}{10 \times 129} \times \cdot 9615 = \cdot 3496 \\
92 \text{ and } 86 &= (1+\cdot 3496) \times \frac{10 \times 129}{17 \times 169} \times \cdot 9615 = \cdot 5827 \\
91 \text{ and } 85 &= (1+\cdot 5827) \times \frac{17 \times 169}{26 \times 224} \times \cdot 9615 = \cdot 7507 \\
90 \text{ and } 84 &= (1+\cdot 7507) \times \frac{26 \times 224}{38 \times 299} \times \cdot 9615 = \cdot 8629
\end{aligned}$$

Whence, the value of the annuity required is equal to ·8629.†

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* Dr. Price has given *incorrect* values for annuities on *lives in general,* according to the table of observations for Sweden. See *Obs. on Rev. Pay.* vol. ii. p. 422. He has taken a *mean* between the value of annuities on male and female lives, which is evidently erroneous.

† In Tables XVII to XXIV at the end of this work, the values are deduced from the probabilities of living amongst males and

§ 35. But the best mode of computing the values of annuities is by means of *logarithms:* and the method of doing this will be sufficiently evident from an inspection of the formula in page 31, and from the principles which have been just laid down. For, since $A_{\prime}B_{\prime}C_{\prime} = (1 + ABC) \times \frac{a\,b\,c}{a_{\prime}b_{\prime}c_{\prime}}(1+\varrho)^{-1}$, it is manifest that $\text{Log.}\,A_{\prime}B_{\prime}C_{\prime} = \text{Log.}\,a + \text{Log.}\,b + \text{Log.}\,c + \text{Log.}\,(1+\varrho)^{-1} + \text{Log.}\,(1+ABC) - [\text{Log.}\,a_{\prime} + \text{Log.}\,b_{\prime} + \text{Log.}\,c_{\prime}]$: whence the calculations, which appear so laborious and intricate when 2 or 3 joint lives are involved, are reduced to the simple operations of addition and subtraction.

§ 36. In calculating the values of annuities according to this method, the following directions should be observed. Begin with the oldest life, C; and write down horizontally on a paper (divided into columns as in the annexed specimen) the logarithms of the number of persons living at the end of n, $(n-1)$, $(n-2)$, &c, years from birth; n denoting the number of years between *birth* and the age of the oldest life in the table of observations. Then, proceeding to the next oldest life B, write down in a similar manner, under the former, the logarithms of the number of persons living

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females *collectively:* but it would be very desirable to have tables of the values of annuities on joint lives, where the ages of the two sexes are kept *distinct* from each other. See the note to the second example in Question IV, Chapter XII.

at the end of $(n-\Delta)$, $(n-\Delta-1)$, $(n-\Delta-2)$, &c, years from birth: Δ denoting the difference of age between B and C. And so on to the next oldest life, according to the number of joint lives required. Add these several perpendicular values together, and write down under each of them the logarithm of $(1+\varrho)^{-1}$. This being done, the subsequent operations become extremely easy, as will sufficiently appear from the following specimen, which shows the method of obtaining the values of annuities on two joint lives of all ages whose difference of age is 10 years; reckoning interest at 4 per cent, and the probabilities of living as at *Northampton.*

Specimen of an easy and expeditious method of calculating the values of two joint lives.
Difference of age 10 *years—Interest* 4 *per cent*—Northampton *Observations.*

	Index to the values in each line.	Ages 95 and 85	Ages 94 and 84	Ages 93 and 83	Ages 92 and 82	Ages 91 and 81	Ages 90 and 80
D	Logs. of 1, 4, 9, 16, 24, 34, &c..........	·0000000	·6020600	·9542425	·2041200	·3802112	5314789
E	Logs. of 145, 180, 234, 289, 346, 406, &c.	·1613080	·2695129	·3692159	·4608978	·5390761	·6085260
F	Sum of the two preceding logarithms	·1613080	·8715729	·3234584	·6650178	·9192873	·1400049
G	Log. of $(1{\cdot}04)^{-1}$ = Log. of ·961538	·9829667	·9829667	·9829667	·9829667	·9829667	·9829667
H	Logs. of the Nat. numbers in S, added to unity	....	·0745970	·1471660	·2060508	·2705572	·3177415
I	Sum of the logarithms in F, G, H, ...	·1443347	·9201366	·4535911	·8560413	·1728112	·4407142
K	Logs. of F in the subsequent *column*	·8715729	·3234584	·6650178	·9192873	·1400049	&c
L	Difference of the two preceding logarithms.	·2727618	·6056782	·7885733	·9307540	·0328063	&c
M	Logs. of $(1{\cdot}04)^{-85}\,(1{\cdot}04)^{-84}\,(1{\cdot}04)^{-83}$ &c	·5521661	·5691994	·5862328	·6032661	·6202995	&c
N	Sum of the logarithms in K and M	·4237390	·8926578	·2512506	·5225534	·7603044	&c,
O	Sum of the logarithms in L and N: or in I and M ..	·6965008	·4983360	·0398239	·4593074	·7931107	&c
P	Nat. numbers to the logarithms in N	26·53011	78·10121	178·3407	333·0837	575·8434	&c
Q	Nat. numbers to the logarithms in O	4·97165	31·50185	109·6032	287·9436	621·0271	&c
R	Sums of the Nat. numbers in P and Q	31·50176	109·60306	287·9439	621·0273	1196·8705	&c
S	Nat. numbers to the logarithms in L which are the values of the an. required ..	·18740	·40335	·61457	·86448	1·07846	&c.

§ 37. In taking out these logarithms from the Tables, the *decimal* parts only are to be retained: and the same observation will apply to the addition and subtraction of the logarithms in the specimen. No error is likely to arise from thus omitting the index, because in the last logarithms resulting from the process (and which are the logarithms of the values of the annuity required in line L) the index when positive never exceeds 1; and when negative it cannot well be mistaken.

In every column, the numbers in the lines D, E, F, G, H, I, K, L, show the process for finding the *value* of the annuity: and are all that are required for *that* purpose. The last line L shows the logarithms of such values: and the natural numbers corresponding to those logarithms (that is, the true values of the annuity required) are set down in the line S. The reason for their being thus kept detached will appear in the sequel; when I come to notice the subsequent lines M, N, O, P, Q, R: and which, for the present, we must suppose not to be in existence.

§ 38. This specimen may be accommodated to the calculation of the values of annuities on *single* lives, by striking out the logarithms in the line E and F. Or, it may be accommodated to the calculation of the values of annuities on any *three* joint lives ABC, if to the logarithms in the lines D and E we add (in order to obtain the line F) the logarithms of the number of persons living at the end of $(n-\delta)$,

$(n-\delta-1)$, $(n-\delta-2)$, &c years from birth: δ denoting the difference between the age of A and C. Hence it is evident that, by this method, the computations of the values of annuities on any *two*, *three*, or *four* joint lives, are nearly as easy as the computations of the values of annuities on any *single* lives: and that the principal labour will consist in taking out the logarithms of the number of persons living at the respective ages; and in finding the natural numbers corresponding to the values in the line L.

§ 39. It is obvious that an error in any one of the numbers will cause an error in *all* the subsequent numbers: and therefore, since the detection of an error at the end of the process might be attended with infinite trouble, it would be a great satisfaction to be enabled to prove the accuracy of the numbers in *each* column, as we go on. A method of doing this has been pointed out by Mr. Morgan, in his *Doctrine of Annuities, &c*, page 58; from which I have deduced the following rule.

§ 40. In the horizontal line M write down the logarithms of $(1+\varrho)^{-(n-\delta-1)}$, $(1+\varrho)^{-(n-\delta-2)}$, $(1+\varrho)^{-(n-\delta-3)}$, &c*: δ denoting the difference of age between the oldest and the youngest life involved. Add these to the logarithms in the line K and set down the sum of them in the line N: add them also to the logarithms in the line I, and set down the sum of these in the line O. Next take out the natural numbers* to the logarithms thus set down in the lines N and O; and place these in the lines P and Q respectively. Then if the natural number in the line Q, in any column, is equal to the sum of the natural numbers in the lines P and Q, in the preceding column, all the values *thus far* have been correctly determined. But, if the first 5 or 6 figures of such number in Q do not agree with the first 5 or 6 figures of the number in the line R, in the preceding column, it will show that there is some inaccuracy in the numbers employed *since the last proof:* and consequently, a revision of the process *from that point* becomes necessary.†

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* The best and most expeditious method of finding the subsequent logarithms of these quantities, after having obtained the logarithm of $(1+\varrho)^{-(n-\delta-1)}$, is to add thereto the logarithm

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of $(1+\varrho)$ for each succeeding year. Thus, in the preceding specimen, the logarithm of $(1+\varrho)^{-(n-\delta-1)}$, or $(1\cdot04)^{-85}$, is equal to ·5521661; and the logarithm of 1·04 is equal to ·0170333: therefore ·0170333 added continually to ·5521661, will give all the subsequent values in the line M.

* In order to take out these logarithms in a proper manner, it might seem necessary to retain the *indices* of the logarithms throughout the whole process: but, after a short trial, the computist will find that he is not likely to be led into any error by omitting them, and that it will save much time and labour if he reject them altogether.

† In revising the process here alluded to, it will be necessary to examine the numbers in the lines N, P, R, S, in the *preceding*

§ 41. Now, though the labour of computing the values of annuities on any lives is somewhat increased by this method of proceeding, yet the additional trouble is more than compensated by the satisfaction which results from a knowledge of the accuracy of the values obtained; as well as by the time and labour which are saved in consequence of detecting every error as it arises.

§ 42. I have entered thus fully into the best method of deducing the values of annuities on single and joint lives, because we are still in want of many accurate tables upon this subject, not only to complete those we are already possessed of; but more particularly some *new* ones founded on the probabilities of life in *this* country as observed amongst annuitants *only*, and also on the probabilities of life as observed in the kingdom *at large*. Such tables would be extremely valuable for general use; but the trouble of calculating them, even with the aid of logarithms, is

column; the numbers in the lines D, E, F, in the *subsequent* column; and the numbers in the lines G, H, I, K, L, M, O, Q, in the column where the proof is found to fail.

Dr. Price observes (*Obs. on Rev. Pay.* vol. i. p. 218) that in calculating the *last* value (that is, when we come to the age of the youngest life in the table of observations) this proof will change into a *new* proof, verifying all the preceding values. This, however, is not the case: the rule which he has given for this purpose is still the *same* proof differently worded; and, *in common with* each proof in its respective column, verifies all the preceding values.

so great that few *individuals* (were they possessed of sufficient *data*, whereon to proceed) would be disposed to undertake so laborious a task: and it unfortunately happens that those *Societies*, whose province it seems more peculiarly to be to patronize an undertaking of that kind, would find little interest in giving the public any additional information on this subject.

§ 43. At the end of the present work I have inserted a collection of all the tables of the values of annuities on single and joint lives, that have hitherto been published*: from which, the computist may select such as appear to relate more particularly to the cases laid before him. Or, he may occasionally make use of them all, according to the condition, situation, and circumstances of the parties concerned. The use and application of these tables will be more fully explained in the last chapter.

COROLLARY III.†

§ 44. By means of the general expression in the problem, for finding the value of an annuity on any given lives, we may determine the value of a *Deferred* life annuity; that is, of an annuity which is not to commence till the end of a given number of years $(=n)$ provided the given lives are then in being, and to continue from that period to the extinction of such given lives. For, let A, B, C, &c, be the given lives; and let $\alpha, \overset{\prime}{\alpha}, \overset{\prime\prime}{\alpha}, \overset{\prime\prime\prime}{\alpha}$, &c; $\beta, \overset{\prime}{\beta}, \overset{\prime\prime}{\beta}, \overset{\prime\prime\prime}{\beta}$, &c; $\gamma, \overset{\prime}{\gamma}, \overset{\prime\prime}{\gamma}, \overset{\prime\prime\prime}{\gamma}$, &c; &c denote the number of persons living, according to any table of observations, at the end of n, $(n+1)$, $(n+2)$, $(n+3)$, &c, years, from the several ages of A, B, C, &c, respectively, as explained in § 23: then will the value of an annuity on the lives ABC&c, be expressed by the following series, $\frac{1}{abc\,\&c} \times \left[\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}\,\&c}{(1+\;)} + \frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}\,\&c}{(1+\varrho)^2} + \frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}\,\&c}{(1+\varrho)^3} + \ldots\ldots \frac{\overset{\prime}{\alpha}\overset{\prime}{\beta}\overset{\prime}{\gamma}\,\&c}{(1+\varrho)^{n+1}} + \frac{\overset{\prime\prime}{\alpha}\overset{\prime\prime}{\beta}\overset{\prime\prime}{\gamma}\,\&c}{(1+\varrho)^{n+2}} + \frac{\overset{\prime\prime\prime}{\alpha}\overset{\prime\prime\prime}{\beta}\overset{\prime\prime\prime}{\gamma}\,\&c}{(1+\varrho)^{n+3}} + \&c\right]$; where the first part is to be continued to n terms, and where the latter part (or that which is deferred) is to be continued to the utmost extent of human life. Consequently this latter part, or the series $\frac{1}{abc\,\&c} \times \left[\frac{\overset{\prime}{\alpha}\overset{\prime}{\beta}\overset{\prime}{\gamma}\,\&c}{(1+\varrho)^{n+1}} + \frac{\overset{\prime\prime}{\alpha}\overset{\prime\prime}{\beta}\overset{\prime\prime}{\gamma}\,\&c}{(1+\varrho)^{n+2}} + \frac{\overset{\prime\prime\prime}{\alpha}\overset{\prime\prime\prime}{\beta}\overset{\prime\prime\prime}{\gamma}\,\&c}{(1+\varrho)^{n+3}} + \&c\right]$ will express the value of an annuity to be *entered on** at the end of n years; and the sum of which I shall denote by $(ABC\&c)^{\text{d}}$.* Now, if we have tables of the values of annuities on single and joint lives of all ages, we may easily deduce the value of this series (without the actual calculation of each separate term) from *the value of an annuity on the same number of lives each* n *years older than the given lives.* For, if this latter value be denoted by $A^n B^n C^n$&c,† we shall have, according to the prin-

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* Tables V to XIII are (by permission of Baron Maseres) taken from his valuable *Treatise on Annuities:* Tables XVI to XXIV, and from XXVII to XLIV are taken from Dr. Price's *Observations on Rev. Pay.* vol. ii. edit. 1803: Tables XLVII and XLVIII are taken from Mr. Simpson's *Treatise on Annuities.* Table XLIX is taken from Mr. Dodson's *Mathematical Repository,* vol. ii: and Table L is taken from Dr. Price's work above mentioned.

† Price, Note (B). Simpson's Sup. Prob. 13.

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* The first payment of which, however, will not take place till the end of the $(n+1)^{st}$ year. See my *Doctrine of Interest and Annuities*, p. 65, note.

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* Care should be taken not to mistake the new character, here introduced, for the *index* of the quantity within the vinculum. In the doctrine of annuities, the notation is, in several instances, different from that of the common analysis of quantities: for instance, the juxtaposition of the letters *ABC* &c, does not denote the multiplication of the quantities represented by those letters, because those letters jointly represent but one quantity; neither does A^x, as used by Mr. Morgan, denote the xth power of an annuity on the life of A. These new methods of notation, after a proper explanation, do not cause any ambiguity; as will be readily felt by those who are acquainted with the Differential and Integral Calculus: and with respect to the symbol here introduced, I would observe that (similar to the like character in the Differential Calculus) it does not altogether represent a quantity, but merely an *operation*. This operation consists in substituting, for the value of an annuity on the given lives within the vinculum, the value of an annuity on the same number of lives each older by the given term, and in multiplying the same by the expectation that the original lives will receive £1 at the end of that term.

† I would here observe that I denote the value of an annuity on a life n years older than A, B, C, &c, by the characters A^n, B^n, C^n, &c: and on two joint lives, each n years older than AB, AC, BC, &c; by the characters A^nB^n, A^nC^n, B^nC^n, &c: and on three joint lives, each n years older than ABC, &c, by $A^nB^nC^n$, &c.

ciples laid down in the problem, $A'B'C'\&c = \frac{1}{\alpha\beta\gamma\,\&c} \times \left[\frac{\overset{\prime}{\alpha}\overset{\prime}{\beta}\overset{\prime}{\gamma}\,\&c}{(1+\varrho)} + \frac{\overset{\prime\prime}{\alpha}\overset{\prime\prime}{\beta}\overset{\prime\prime}{\gamma}\,\&c}{(1+\varrho)^2} + \frac{\overset{\prime\prime\prime}{\alpha}\overset{\prime\prime\prime}{\beta}\overset{\prime\prime\prime}{\gamma}\,\&c}{(1+\varrho)^3} + \&c\right]$: therefore, multiplying both sides of this equation by $\frac{\alpha\beta\gamma\,\&c}{abc\,\&c}(1+\varrho)^{-n}$, we produce $A'B'C'\&c \times \frac{\alpha\beta\gamma\,\&c}{abc\,\&c}(1+\varrho)^{-n} = (ABC\&c)^{d}$: whence the following general rule.

§ 45. *Find the value of an annuity on the same number of lives, each as many years older than the given lives as are equal to the number of years during which the annuity is deferred; find also the expectation* of the given lives receiving £1 at the end of that term: the product of these two quantities will be the value required.*†

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* The *expectation* of receiving any sum of money at the end of any given term is equal to the present value of such sum multiplied by the probability that the given lives will continue in being so long: that is, equal to $\frac{\alpha\beta\gamma\,\&c}{abc\,\&c} \times (1+\varrho)^{-n}$. See Chap. I, § 5 and 6.

† The periods for which any annuity may be deferred, and the variety of ages with which the same may be combined, would render it an endless labour to compute a table of the value of deferred life annuities to suit every case. Mr. Baron Maseres has given a partial table of this kind, which shows the present value of annuities, on single lives, to commence at the end of *thirty* years, adapted to every age from 3 to 64 years, founded on M. De Parcieux's tables of observations, and at the several rates of $3\frac{1}{2}$, 4, $4\frac{1}{2}$, and 5 per cent interest. See his *Doctrine of Annuities*, p. 283 to 287.

For examples of the use and application of this corollary, see Question VI in Chapter XII.

COROLLARY IV.*

§ 46. By the help of the series in the last corollary we may determine the value of a *Temporary* life annuity; that is, of an annuity which is to commence immediately, but to continue only for a given number of years ($=n$) which is less than that to which it is possible that the given life or lives may extend†, and then to cease. For, supposing every thing to remain as in the last corollary, it is evident that the first n terms of the series there given, or $\frac{1}{abc\,\&c} \times \left[\frac{\overset{\prime}{a}\,\overset{\prime}{b}\,\overset{\prime}{c}\,\&c}{(1+\varrho)} + \frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{b}\,\overset{\prime\prime}{c}\,\&c}{(1+\varrho)^2} + \frac{\overset{\prime\prime\prime}{a}\,\overset{\prime\prime\prime}{b}\,\overset{\prime\prime\prime}{c}\,\&c}{(1+\varrho)^3} + \ldots\ldots\ldots \frac{\alpha\,\beta\,\gamma\,\&c}{(1+\varrho)^n}\right]$ will be the value of the temporary annuity required, and which I shall denote by $(ABC\&c)^{t}$.‡ Now by means of the tables above mentioned, we may easily deduce the value of this series without the actual enumeration of its several terms: for, it is evident on inspection that the value of the first n terms of the series in the last corollary is equal to the value of the whole series *minus* the sum of all the terms after the n^{th} term. But the sum of all the terms after the n^{th} term has been found, by the last corollary, to be equal to $(ABC\&\text{c})^{\text{d}}$: consequently the sum of the first n terms is equal to $ABC\&\text{c} - (ABC\&\text{c})^{\text{d}}$; whence the following rule.

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* Simpson's Sup. Prob. 6. De Moivre, Prob. 26. Morgan, Prob. 4. Dodson, vol. iii. Ques. 2, 7, and 10.

† This is always understood in cases of this kind: and it may be here useful to remark that the term during which it is possible any given life may be prolonged is equal to the difference between the age of that life, and the age of the oldest life in the table of observations. In the case of joint lives, it is equal to the difference between the oldest of such joint lives, and the age of the oldest life in the table of observations.

‡ Agreeably to what has been said in the note in page 43, the reader must be careful not to mistake this character for the *index*

§ 47. *Find the value of an annuity on the given lives deferred during the given period, and deduct this value from the value of an annuity on the given lives: the difference will be the value required.*

For examples of the use and application of this corollary, see Question VII in Chapter XII.

COROLLARY V.

§ 48. By a similar method of reasoning we may find the value of a *Deferred and Temporary* life annuity; that is, of an annuity which is not to commence till the end of a given number of years $(= n)$,

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of the quantity within the vinculum. For, the symbol here introduced denotes also rather an *operation* to be performed: and which operation consists in substituting, for the original quantity, the *difference* between such quantity and the new one found by the method laid down in the note above alluded to.

and then to continue only during another period ($= \nu$), which is less than that to which it is possible the given lives may *then* be prolonged. And, from what has been said in the third corollary, the truth of the following rule will be evident.

§ 49. *From the value of an annuity on the given lives, deferred for the term* n, *subtract the value of the same annuity on such lives, deferred for the term* ($n + \nu$)*: the difference will be the sum of the required* ν *terms of the series, or the value of the annuity proposed.*

COROLLARY VI.*

§ 50. If the annuity is to be enjoyed for a term *certain* ($= n$), and after that during the continuance of any given lives, its value will be expressed by the following series $\frac{1}{(1+\varrho)} + \frac{1}{(1+\varrho)^2} + \frac{1}{(1+\varrho)^3} + \ldots\ldots \frac{1}{(1+\varrho)^n} +$ $\frac{1}{a\,b\,c\,\&c} \times \left[\frac{\alpha'\beta'\gamma'\,\&c}{(1+\varrho)^{n+1}} + \frac{\alpha''\beta''\gamma''\,\&c}{(1+\varrho)^{n+2}} + \frac{\alpha'''\beta'''\gamma'''\,\&c}{(1+\varrho)^{n+3}} + \&c \right]$. But, the first n terms of this series, or $\frac{1}{(1+\varrho)} + \frac{1}{(1+\varrho)^2}$ $+ \frac{1}{(1+\varrho)^3} + \ldots\ldots\ldots \frac{1}{(1+\varrho)^n}$, are equal to $\frac{1-(1+\varrho)^{-n}}{\varrho}$, or to the present value of an annuity of £1 per annum for n years; and the remaining part of the series is, by the third corollary, equal to $(ABC\&c)^{d}$: whence the following rule.

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* Morgan, Prob. 15.

§ 51. *To the present value of an annuity certain for the given term, add the value of an annuity on the given lives deferred for that term: the sum will be the answer required.*

For examples of the use and application of this corollary, see Question XXII in Chapter XII.

COROLLARY VII.

§ 52. If, in the general expression for the value of an annuity on a single life A, we suppose ρ to vanish, or that money does not bear any interest; then will such expression be reduced to the series $\frac{1}{a}(\overset{\prime}{a}+\overset{\prime\prime}{a}+\overset{\prime\prime\prime}{a}+\&c)$, and which is the value of an annuity on the life A considered as a *yearly* annuitant without interest of money: whence the following rule. Divide the sum of all the living, at every age after the age of the given life, by the number of persons living at that age; and the quotient will be the value required.

In a subsequent chapter I shall show that we must add $\frac{1}{4}$ or $\frac{3}{8}$ to the value thus found, in order to obtain the value of a similar annuity on the life A considered as a *half yearly* or *quarterly* annuitant*: but when the life is considered as a *momently* annuitant, we must add $\frac{1}{2}$ to such value. In this last case the required value coincides with the sums of the probabilities that such life will attain to the end of the 1st, 2nd, 3rd, &c, *moments* from the present time to the end of its possible existence: and is the same with what is denominated, by writers on this subject, the *Expectation of Life*; or the number of years which, taking lives of the same age one with another, any one of those lives may be considered as sure of enjoying; those who live *beyond* that period, enjoying as much *more* in proportion to their number, as those who *fall short* of it enjoy *less*. Consequently the rule for finding the Expectation of Life will be as follows.

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* See Chapter X.

§ 53. *Divide the sum of all the living, at every age after the age of the given life, by the number of persons living at that age: half unity added to the quotient will be the value required.*

§ 54. As we may sometimes have occasion to determine this value in order to compare the probabilities of life, or the values of annuities, according to different tables of observations, as well as for various purposes of approximation to be explained hereafter, I have here inserted the rule for finding such values according to any table of observations. But in Table II at the end of the present treatise, I have given the Expectation of Life for every *fifth* year as deduced from the several tables of observations mentioned in page 9: and in

Tables IV, XV, XXVI and XLVI, I have given the Expectations of Life for *every* year, as deduced from the tables of observations immediately preceding*.

PROBLEM II.†

§ 55. To find the value of an annuity granted upon the longest of any number of lives; that is, for as long as any one of them is in existence.

SOLUTION.

Let A, B, C, &c, be the lives upon which the annuity is granted, and let the probability of each life continuing 1, 2, 3, &c, years, be as denoted in § 23: then, the probability that some one or other of these will live to the end of the first year will (by § 26) be expressed by $1 - \left(1 - \frac{\acute{a}}{a}\right) \times \left(1 - \frac{\acute{b}}{b}\right) \times \left(1 - \frac{\acute{c}}{c}\right) \times$ &c $= \frac{\acute{a}}{a} + \frac{\acute{b}}{b} + \frac{\acute{c}}{c} +$ &c $- \frac{\acute{a}\acute{b}}{ab} - \frac{\acute{a}\acute{c}}{ac} - \frac{\acute{b}\acute{c}}{bc} -$ &c $+ \frac{\acute{a}\acute{b}\acute{c}}{abc} +$ &c; which, being multiplied by $(1+\rho)^{-1}$, or the present value of £1 certain to be received at the end of one year, will give the present value of the first year's rent, or the *expectation* of receiving such sum on the contingency that any one of the lives continues to the end of the first year. In like manner, since the probability that some one or other of the lives will live to the end of the second year is $1-\left(1-\frac{\overset{\prime\prime}{a}}{a}\right)\times\left(1-\frac{\overset{\prime\prime}{b}}{b}\right)\times\left(1-\frac{\overset{\prime\prime}{c}}{c}\right)\times \&c=\frac{\overset{\prime\prime}{a}}{a}+\frac{\overset{\prime\prime}{b}}{b}+\frac{\overset{\prime\prime}{c}}{c}+\&c-\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{ab}-\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{c}}{ac}-\frac{\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{bc}-\&c+\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}+\&c$; it follows that if this expression be multiplied by $(1+\varrho)^{-2}$, or the present value of £1 certain to be received at the end of two years, it will give the present value of the second year's rent, or the *expectation* of receiving such sum on the contingency that some one or other of the lives will live to the end of the second year. By the same method of reasoning it will be found that $1-\left(1-\frac{\overset{\prime\prime\prime}{a}}{a}\right)\times\left(1-\frac{\overset{\prime\prime\prime}{b}}{b}\right)\times\left(1-\frac{\overset{\prime\prime\prime}{c}}{c}\right)\times \&c=\frac{\overset{\prime\prime\prime}{a}}{a}+\frac{\overset{\prime\prime\prime}{b}}{b}+\frac{\overset{\prime\prime\prime}{c}}{c}+\&c-\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{ab}-\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{c}}{ac}-\frac{\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{bc}-\&c+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}+\&c$, being multiplied by $(1+\varrho)^{-3}$, will give the present value of the third year's rent, or the *expectation* of receiving such sum on the contingency that some one or other of the lives will live to the end of the third year. And so on for all the subsequent years, to the utmost extent of human life: the sum of all which expectations,* or the series

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* I would here observe that, according to the hypothesis of M. De Moivre, which will be explained hereafter, the expectation of any single life is equal to *half the complement* of that life: consequently the Complement of any life is equal to twice the Expectation of that life. This mode of expression is sometimes used, even when speaking of values *deduced from real observations*.

† Simpson, Prob. 2. De Moivre, Prob. 4 and 5. Dodson, vol. ii. Ques. 76 to 86: vol. iii. Ques. 14.

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* When the ages of the given lives differ, the number of terms composing those several series will also differ: but, I would here observe, once for all, that in every case of combined lives the terms must be continued to the extinction of the oldest life in-

$$\frac{1}{(1+\varrho)}\left[\frac{\overset{\prime}{a}}{a}+\frac{\overset{\prime}{b}}{b}+\frac{\overset{\prime}{c}}{c}+\&c-\frac{\overset{\prime}{a}\,\overset{\prime}{b}}{a\,b}-\frac{\overset{\prime}{a}\,\overset{\prime}{c}}{a\,c}-\frac{\overset{\prime}{b}\,\overset{\prime}{c}}{b\,c}-\&c+\frac{\overset{\prime}{a}\,\overset{\prime}{b}\,\overset{\prime}{c}}{a\,b\,c}+\&c\right]+$$

$$\frac{1}{(1+\varrho)^2}\left[\frac{\overset{\prime\prime}{a}}{a}+\frac{\overset{\prime\prime}{b}}{b}+\frac{\overset{\prime\prime}{c}}{c}+\&c-\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{b}}{a\,b}-\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{c}}{a\,c}-\frac{\overset{\prime\prime}{b}\,\overset{\prime\prime}{c}}{b\,c}-\&c+\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{b}\,\overset{\prime\prime}{c}}{a\,b\,c}+\&c\right]+$$

$$\frac{1}{(1+\varrho)^3}\left[\frac{\overset{\prime\prime\prime}{a}}{a}+\frac{\overset{\prime\prime\prime}{b}}{b}+\frac{\overset{\prime\prime\prime}{c}}{c}+\&c-\frac{\overset{\prime\prime\prime}{a}\,\overset{\prime\prime\prime}{b}}{a\,b}-\frac{\overset{\prime\prime\prime}{a}\,\overset{\prime\prime\prime}{c}}{a\,c}-\frac{\overset{\prime\prime\prime}{b}\,\overset{\prime\prime\prime}{c}}{b\,c}-\&c+\frac{\overset{\prime\prime\prime}{a}\,\overset{\prime\prime\prime}{b}\,\overset{\prime\prime\prime}{c}}{a\,b\,c}+\&c\right]+$$

$$\qquad\&c\qquad\qquad\qquad\&c\qquad\qquad\qquad\qquad\&c$$

will be the total present value of the annuity required.

§ 56. But, the first collateral column in this general expression denotes the value of an annuity on the life A; the second denotes the value of an annuity on the life B; the third denotes the value of an annuity on the life C; &c. In like manner, the fourth, fifth and sixth collateral columns denote the values of an annuity on the joint lives AB, AC, BC, &c respectively: and so on. Whence, if we substitute for these respective values the characters mentioned in Prob. I, cor. 1, the whole expression, or the sum of all the terms in the above series, will become equal to $A+B+C+\&c-AB-AC-BC-\&c+ABC+$ &c: consequently, the value of an annuity, to continue as long as any one of the given lives is in existence, is equal to the sum of the values of an annuity on all the single lives, *minus* the sum of the values of an annuity on all the joint lives combined two and two,

volved in the combination; or universally, *each* series must be continued till the terms vanish, which will happen when either of the lives involved in such series has arrived at the extremity of human life.

plus the sum of the values of an annuity on all the joint lives combined three and three, *minus* the sum of the values of an annuity on all the joint lives combined four and four; and so on. Therefore, when the values of an annuity on the single and joint lives are given, the value of an annuity on the longest life may be very easily determined: and, for the sake of a more convenient reference, I shall take L to denote the value found by this rule; the *number* of lives, which it is intended to represent, being always explained when the character is used.

For examples of the use and application of this problem, see Questions VIII and IX in Chapter XII.

COROLLARY I.[*]

§ 57. If the lives are all equal, or of the same age A, and their number be represented by n; then the probability that some one or other of such lives will continue to the end of the first, second, third, &c, years respectively, will be severally denoted by $1-\left(1-\frac{\overset{\prime}{a}}{a}\right)^n$, $1-\left(1-\frac{\overset{\prime\prime}{a}}{a}\right)^n$, $1-\left(1-\frac{\overset{\prime\prime\prime}{a}}{a}\right)^n$, &c. Let these quantities be severally expanded by means of the binomial theorem, in order to reduce them to simple terms, and then be respectively multiplied into the present value of £1 due at the end of those years; the sum of them, or the series

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[*] Dodson, vol. ii. Ques. 78, 85 and 86:—Vol. iii. Ques. 11.

$$\begin{aligned}
&\frac{1}{(1+\varrho)}\left[n.\frac{\overset{\prime}{a}}{a}-\frac{n.(n-1)}{2}.\frac{\overset{\prime}{a}\,\overset{\prime}{a}}{a\,a}+\frac{n.(n-1)}{2}.\frac{n-2}{3}.\frac{\overset{\prime}{a}\,\overset{\prime}{a}\,\overset{\prime}{a}}{a\,a\,a}-\&c\right]+\\
&\frac{1}{(1+\varrho)^2}\left[n.\frac{\overset{\prime\prime}{a}}{a}-\frac{n.(n-1)}{2}.\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{a}}{a\,a}+\frac{n.(n-1)}{2}.\frac{n-2}{3}.\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{a}\,\overset{\prime\prime}{a}}{a\,a\,a}-\&c\right]+\\
&\frac{1}{(1+\varrho)^3}\left[n.\frac{\overset{\prime\prime\prime}{a}}{a}-\frac{n.(n-1)}{2}.\frac{\overset{\prime\prime\prime}{a}\,\overset{\prime\prime\prime}{a}}{a\,a}+\frac{n.(n-1)}{2}.\frac{n-2}{3}.\frac{\overset{\prime\prime\prime}{a}\,\overset{\prime\prime\prime}{a}\,\overset{\prime\prime\prime}{a}}{a\,a\,a}-\&c\right]+\\
&\qquad\quad \&c \qquad\qquad\qquad \&c \qquad\qquad\qquad\qquad\qquad \&c
\end{aligned}$$

will be the total present value of the annuity required.

§ 58. Whence (if we take AA, AAA, &c, to denote the value of an annuity on two, three, &c, equal joint lives) it is manifest that the value of an annuity on the longest of any number of lives, all equal to A, will be $nA-\frac{n.(n-1)}{2}AA+\frac{n.(n-1)}{2}.\frac{n-2}{3}AAA-\&c$. Therefore, if the number of lives be two, this expression will become $2A-AA$; if three, it will become $3A-3AA+AAA$; if four, it will become $4A-6AA+4AAA-AAAA$; and so on.

For examples of the use and application of this corollary, see Question VIII in Chapter XII.

COROLLARY II.*

§ 59. If we wish to determine the value of an annuity on the longest of any number of lives†, *Deferred* for a given term ($= n$), it will be evident (from what has been said in Prob. I, cor. 3) that the several perpendicular series in page 52 must not commence till after the n^{th} term; and thence be continued to the utmost extent of human life. Consequently, the required value of such deferred annuity will be equal to $(A)^{\text{d}} + (B)^{\text{d}} + (C)^{\text{d}} - (AB)^{\text{d}} - (AC)^{\text{d}} - (BC)^{\text{d}} + (ABC)^{\text{d}}$: an expression which I shall, for the sake of a more convenient reference, denote by $(L)^{\text{d}}$. Now, if for each of these several quantities we substitute its corresponding value, agreeably to the principles laid down in page 44, the formula denoted by $(L)^{\text{d}}$ will become equal to $A^{\circ} \times \frac{\alpha}{a}(1+\varrho)^{-n} + B^{\circ} \times \frac{\beta}{b}(1+\varrho)^{-n} + C^{\circ} \times \frac{\gamma}{c}(1+\varrho)^{-n} - A^{\circ}B^{\circ} \times \frac{\alpha\beta}{ab}(1+\varrho)^{-n} - A^{\circ}C^{\circ} \times \frac{\alpha\gamma}{ac}(1+\varrho)^{-n} - B^{\circ}C^{\circ} \times \frac{\beta\gamma}{bc}(1+\varrho)^{-n} + A^{\circ}B^{\circ}C^{\circ} \times \frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n}$: which is an expression more convenient for practice; and from either of which we deduce the following rule.

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* Dodson, vol. iii. Ques. 13 and 15.

† The investigations in the present corollary are confined to the case of three lives: but it is easy to apply the same method of reasoning, *whatever* be the number of lives involved.

§ 60. *Substitute the values of deferred annuities on the single and joint lives, in the general rule in the problem, instead of the values of annuities on the whole continuance of those lives; and proceed with these substituted values according to the directions given in such rule: the result will be the answer required.*

For examples of the use and application of this corollary, see Question XI in Chapter XII.

COROLLARY III.

§ 61. If an annuity on the longest of any number of lives, deferred for any given term, depends on the *joint* continuance of all those lives to the end of that term; the value of it will be equal to $L^{\circ} \times \frac{\alpha\beta\gamma}{abc}(1+\rho)^{-n}$: that is, equal to the value of an annuity on the longest of the same number of lives, each older by the given term than the given lives,* multiplied by the expectation that the *joint* lives shall receive £1 at the end of that term: and this case must be carefully distinguished from the one mentioned in the preceding corollary.

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* Upon the same principle that I have taken $(ABC\&c)^{d}$ and $(ABC\&c)^{t}$, in Prob. I, cor. 3 and 4, to denote the value of a deferred and temporary annuity on the *joint* lives ABC&c, I shall take $(L)^{d}$ and $(L)^{t}$ as the expressions for the value of a deferred and temporary annuity on the *longest* of those lives. Having also taken $A^{\circ}B^{\circ}C^{\circ}$ *&c* to denote the value of an annuity on the same number of *joint* lives, each older by the given term than the lives ABC &c, I shall in like manner take L° as the expression for the value of a similar annuity on the *longest* of such lives. This method of notation will be found very convenient in many of the subsequent problems: and, being very simple in its principle, will not cause any confusion or intricacy from its novelty. The *number* of lives denoted by L will be always mentioned whenever the character is used: as I have already observed in page 53.

For examples of the use and application of this corollary, see the Scholium to Question XI in Chapter XII.

COROLLARY IV.

§ 62. Having found, by means of the second corollary here given, the value of a deferred annuity on the longest of any number of lives, we may easily determine the value of a *Temporary** annuity on the longest of such lives: it being nothing more than the difference between the value of such deferred annuity and the value of a similar annuity on the whole continuance of the lives, as found by the problem. That is, the formula $L-(L)^{\mathrm{d}}$ will in all cases denote the value of such temporary annuity: similar to what takes place with respect to joint lives, as already explained in page 46.

For examples of the use and application of this corollary, see Question XII in Chapter XII.

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* That is, of an annuity which is to commence immediately, but to continue only during a given number of years ($= n$) which is less than that to which it is possible *either* of the given lives may extend. Therefore such term must not be greater than the difference between the age of the *youngest* life involved, and the age of the oldest life in the table of observations: for, in such case, the value is found by the Problem. See what has been said respecting single and joint lives in the note in page 45.

PROBLEM III.[*]

§ 63. To find the value of an annuity granted upon any number of lives, but to continue only as long as any number ($=n$) of them are in being together.

SOLUTION.

Let the lives on which the annuity is granted be A, B, C, &c: and the probability of each life continuing 1, 2, 3, &c, years be as denoted in § 23. Now if we confine this case to that of an annuity granted upon three lives, and to continue as long as any *two* of them (viz. AB, AC, or BC) are in being together[†], it is evident that the chance of the annuity being received in any one year will depend upon either of these four different events: 1° that all the lives continue in being together to the end of that year, the probability of which in the first year is $\frac{\acute{a}\,\acute{b}\,\acute{c}}{a\,b\,c}$: 2° that A and B are then alive and C dead, the probability of which in the same year is $\frac{\acute{a}\,\acute{b}}{a\,b}\times\left(1-\frac{\acute{c}}{c}\right)$: 3° that A and C are then alive and B dead, the probability of which in the same year is $\frac{\acute{a}\,\acute{c}}{a\,c}\times\left(1-\frac{\acute{b}}{b}\right)$: 4° that

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[*] Simpson, Prob. 3. Morgan, Prob. 7. Dodson, vol. ii. Ques. 87.

[†] All the cases of *two* lives, and likewise the *remaining cases of three lives*, may be solved by means of the preceding problems.

B and C are then alive and A dead, the probability of which in the same year is $\frac{\overset{\prime}{b}\overset{\prime}{c}}{bc}\times\left(1-\frac{\overset{\prime}{a}}{a}\right)$. The sum of all these chances, therefore, or $\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}+\frac{\overset{\prime}{a}\overset{\prime}{b}}{ab}\times\left(1-\frac{\overset{\prime}{c}}{c}\right)+\frac{\overset{\prime}{a}\overset{\prime}{c}}{ac}\times\left(1-\frac{\overset{\prime}{b}}{b}\right)+\frac{\overset{\prime}{b}\overset{\prime}{c}}{bc}\times\left(1-\frac{\overset{\prime}{a}}{a}\right)=\frac{\overset{\prime}{a}\overset{\prime}{b}}{ab}+\frac{\overset{\prime}{a}\overset{\prime}{c}}{ac}+\frac{\overset{\prime}{b}\overset{\prime}{c}}{bc}-\frac{2\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}$, being multiplied by $(1+\varrho)^{-1}$ will give the present value of the first year's rent, or the *expectation* of receiving such sum on the contingency that any two of the original lives will outlive the first year. By a similar method of reasoning it will be found that $\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{ab}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{c}}{ac}+\frac{\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{bc}-\frac{2\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}$, multiplied by $(1+\varrho)^{-2}$, will give the present value of the second year's rent, or the *expectation* of receiving such sum on the contingency that any two of the original lives will outlive the second year. Also that $\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{ab}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{c}}{ac}+\frac{\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{bc}-\frac{2\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}$, multiplied by $(1+\varrho)^{-3}$, will give the present value of the third year's rent; and so on, for all the subsequent years to the utmost extremity of human life; the sum of all which expectations, or the series

$$(1+\varrho)^{-1}\times\left[\frac{\overset{\prime}{a}\overset{\prime}{b}}{ab}+\frac{\overset{\prime}{a}\overset{\prime}{c}}{ac}+\frac{\overset{\prime}{b}\overset{\prime}{c}}{bc}-\frac{2\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}\right]+$$

$$(1+\varrho)^{-2}\times\left[\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{ab}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{c}}{ac}+\frac{\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{bc}-\frac{2\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}\right]+$$

$$(1+\varrho)^{-3}\times\left[\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{ab}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{c}}{ac}+\frac{\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{bc}-\frac{2\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}\right]+$$

&c &c

will be the total present value of the annuity required.

§ 64. But, if we take the value of each perpendicular series, as in the last problem, and substitute the characters given in Prob. I, cor. 1, the above expression will become $AB + AC + BC - 2ABC$: whence the following rule for this particular case. From the sum of the values of an annuity on each pair of joint lives, take twice the value of an annuity on the three joint lives; the difference will be the required value of an annuity to continue as long as any two of the lives are in being together.

For examples of the use and application of this problem, see Question X in Chapter XII.

§ 65. By a similar method of proceeding we might ascertain the value of an annuity granted on four lives, but to continue only as long as any *two* or *three* of them are in being together (the other cases of such lives being already solved by the two preceding problems); and, in general, the value of an annuity on any number of lives, to continue only as long as any number of them are in being together. As it rarely happens, however, that more than three lives are concerned in any practical cases, I shall not trouble the reader with the steps of the process, but shall merely state the result of the investigation in one general formula, which will comprehend in one view all the possible cases mentioned in the three problems here given*. Let the sum of the values of an annuity on every n joint lives be denoted by S; on every $(n+1)$ joint lives, by $\overset{\text{I}}{S}$; on every $(n+2)$ joint lives, by $\overset{\text{II}}{S}$; on every $(n+3)$ joint lives, by $\overset{\text{III}}{S}$; &c:* then will the value of an annuity granted upon any number of lives, to continue only as long as any n of them are in being together, be expressed by $S - n.\overset{\text{I}}{S} + \frac{n}{1}.\frac{n+1}{2}.\overset{\text{II}}{S} - \frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}.\overset{\text{III}}{S} + \&c.$

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* It will be readily seen that Problem I and II are only particular cases of this third problem.

COROLLARY I.

§ 66. If the annuity is not to continue during the whole period of the given lives, but is either *Deferred* or *Temporary*, we must pursue the same method of reasoning which has been adopted in Prob. II, cor. 2 and 4: whereby it will appear that the value of a deferred annuity, to continue as long as any two out of three given lives are in being, will be expressed by $(AB)^{\text{d}} + (AC)^{\text{d}} + (BC)^{\text{d}} - 2(ABC)^{\text{d}}$. Therefore if we substitute the values of deferred annuities on the given lives (in the formula in § 64) instead of the values of annuities on the whole continuance of those lives, we shall obtain the value of the deferred annuity depending on the contingency alluded to in the problem.

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* That is, on every combination which can be made of the given lives, by combining n, $(n+1)$, $(n+2)$, $(n+3)$, &c, lives together at a time.

And this value, being subtracted from the value of a similar annuity on the whole continuance of the lives, will give the value of a similar *Temporary* annuity on any two out of three lives.

COROLLARY II.

§ 67. If such deferred annuity however depends on the *joint* continuance of all the given lives to the end of the given term, the value of it will be equal to the value of a similar annuity on the same number of lives each older by the given term than such lives, multiplied by the expectation that the *joint* lives shall receive £1 at the end of that term: and this case must be carefully distinguished from that mentioned in the first part of the preceding corollary.

CHAPTER III.

On Reversions.

§ 68. A REVERSIONARY life annuity is a term applied to such periodical sums of money, depending on any given lives, as are not payable till after a given term, or till after the extinction of any other given lives. Of the former kind are all *deferred* life annuities, mentioned in Prob. I, cor. 3; but it is my intention now to treat only of the latter kind: and I would here observe, that I shall continue to designate the former by the title of Deferred life annuities, applying the term *Reversionary* life annuities to such life annuities only as are not to be enjoyed till after the extinction of some other life. The several cases relating to this subject may be comprised in the four following problems.

PROBLEM IV.[*]

§ 69. To find the value of an annuity depending on any number of joint lives A B C &c, after the extinction of any number of other joint lives P Q R &c.[†]

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[*] Simpson, Prob. 5.

[†] The lives P, Q, R, are said to be in *possession*; in opposition to A, B, C, which are said to be in *reversion*.

SOLUTION.

Let the probability of the joint lives A B C* continuing 1, 2, 3, &c, years, be respectively denoted by $\frac{a'b'c'}{abc}$, $\frac{a''b''c''}{abc}$, $\frac{a'''b'''c'''}{abc}$, &c, as in § 24; and let the probability of the joint lives P Q R continuing 1, 2, 3, &c, years be denoted by $\frac{p'q'r'}{pqr}$, $\frac{p''q''r''}{pqr}$, $\frac{p'''q'''r'''}{pqr}$, &c, respectively. Now the chance, which the joint lives A B C have of receiving the annuity in any one year, will depend upon their living to the end of that year and on the joint lives P Q R becoming extinct before the end of that period. The probability of this event happening in the first year is $\frac{a'b'c'}{abc}\left(1-\frac{p'q'r'}{pqr}\right)$; which being multiplied by $(1+\varrho)^{-1}$ will give the present value of the first year's rent, or the expectation of receiving such sum at the end of the first year. By a similar method of reasoning it will be found that $\frac{a''b''c''}{abc}\left(1-\frac{p''q''r''}{pqr}\right)$, multiplied by $(1+\varrho)^{-2}$ will denote the present value of the second year's rent; and that $\frac{a'''b'''c'''}{abc}\left(1-\frac{p'''q'''r'''}{pqr}\right)$, multiplied by $(1+\varrho)^{-3}$, will denote the present value of the third year's rent; and so on for all the subsequent years to the utmost extent of human life: the sum of all which terms, or the series

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* In the solution of this and the following problems, I shall confine the case to *three* lives only, in order to avoid confusion by the constant repetition of the character &c. From what has preceded, it will plainly appear that the conclusions which are drawn from this assumption will be true, whatever be the number of lives involved in the process.

$$\frac{1}{(1+\varrho)}\left[\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}-\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}\overset{\prime}{p}\overset{\prime}{q}\overset{\prime}{r}}{abcpqr}\right]+$$
$$\frac{1}{(1+\varrho)^2}\left[\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}-\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}\overset{\prime\prime}{p}\overset{\prime\prime}{q}\overset{\prime\prime}{r}}{abcpqr}\right]+$$
$$\frac{1}{(1+\varrho)^3}\left[\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}-\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}\overset{\prime\prime\prime}{p}\overset{\prime\prime\prime}{q}\overset{\prime\prime\prime}{r}}{abcpqr}\right]+$$
$$\&c \qquad \&c,$$

will be the total present value of the annuity required. But the sum of these two perpendicular series is evidently equal to $ABC - ABCPQR$.

§ 70. Whence it follows that if we subtract the value of an annuity on all the joint lives, from the value of an annuity on the joint lives in reversion; the remainder will be the value of the reversionary annuity required.

PROBLEM V.*

§ 71. To find the value of an annuity depending on any number of joint lives A B C &c, after the extinction of the longest of any number of other lives P, Q, R, &c.

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* Simpson, Prob. 6.

SOLUTION.

Let the probabilities of the given lives continuing 1, 2, 3, &c, years be denoted by the same characters as in the last problem. Now the chance, which the joint lives A B C have of receiving the annuity at the end of any one year, will depend upon their living to the end of that year, and on all the lives P, Q, R, becoming extinct before the end of that period. The probability of this event happening in the first year is $\frac{a'b'c'}{abc}\left(1-\frac{p'}{p}\right)\times\left(1-\frac{q'}{q}\right)\times\left(1-\frac{r'}{r}\right)$[*], which being multiplied by $(1+\varrho)^{-1}$ will give the present value of the first year's rent. In like manner, $\frac{a''b''c''}{abc}\left(1-\frac{p''}{p}\right)\times\left(1-\frac{q''}{q}\right)\times\left(1-\frac{r''}{r}\right)$ multiplied by $(1+\varrho)^{-2}$, and $\frac{a'''b'''c'''}{abc}\left(1-\frac{p'''}{p}\right)\times\left(1-\frac{q'''}{q}\right)\times\left(1-\frac{r'''}{r}\right)$ multiplied by $(1+\varrho)^{-3}$, will give the present value of the second and third year's rents respectively: and so on for all the subsequent years, to the utmost extent of human life; the sum of all which terms will be the present value of the annuity required. But these expressions, reduced to their simplest terms, are equal to the series

$$\frac{a'b'c'}{abc(1+\varrho)}\left[1-\frac{p'}{p}-\frac{q'}{q}-\frac{r'}{r}+\frac{p'q'}{pq}+\frac{p'r'}{pr}+\frac{q'r'}{qr}-\frac{p'q'r'}{pqr}\right]+$$

$$\frac{a''b''c''}{abc(1+\varrho)^2}\left[1-\frac{p''}{p}-\frac{q''}{q}-\frac{r''}{r}+\frac{p''q''}{pq}+\frac{p''r''}{pr}+\frac{q''r''}{qr}-\frac{p''q''r''}{pqr}\right]+$$

$$\frac{a'''b'''c'''}{abc(1+\varrho)^3}\left[1-\frac{p'''}{p}-\frac{q'''}{q}-\frac{r'''}{r}+\frac{p'''q'''}{pq}+\frac{p'''r'''}{pr}+\frac{q'''r'''}{qr}-\frac{p'''q'''r'''}{pqr}\right]+$$

&c &c &c,

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[*] See Chapter I, § 9.

the sum of which is evidently equal to $ABC - ABCP - ABCQ - ABCR + ABCPQ + ABCPR + ABCQR - ABCPQR$.

§ 72. Whence it follows that the value of a reversionary annuity on any number of joint lives ABC &c, after the extinction of the longest of any number of other lives P, Q, R, &c, is equal to the value of an annuity on all the joint lives in reversion; *minus* the sum of the values of an annuity on all the joint lives, arising from the combination of all the joint lives in reversion with each ONE of the other lives; *plus* the sum of the values of an annuity on all the joint lives, arising from the combination of all the joint lives in reversion with each TWO of the other lives; *minus* the sum of the values of an annuity on all the joint lives, arising from the combination of all the joint lives in reversion with each THREE of the other lives; and so on.

PROBLEM VI.*

§ 73. To find the value of an annuity on the longest of any number of lives A, B, C, &c, after the extinction of any number of joint lives P Q R &c.

SOLUTION.

Let the probabilities of the given lives continuing 1, 2, 3, &c, years be denoted by the same characters as in the preceding problems. Now, the chance which any one of the lives A, B, C, has of receiving the annuity at the end of any one year, will depend upon either of them living to the end of that year, and on the extinction of the joint lives P Q R before the end of that period. The probability of this happening in the first year is $\left[1-\left(1-\frac{\overset{\prime}{a}}{a}\right)\times\left(1-\frac{\overset{\prime}{b}}{b}\right)\times\left(1-\frac{\overset{\prime}{c}}{c}\right)\right]\times\left(1-\frac{\overset{\prime}{p}\overset{\prime}{q}\overset{\prime}{r}}{pqr}\right)$, which being multiplied by $(1+\varrho)^{-1}$ will give the present value of the first year's rent. In like manner it will be found that $\left[1-\left(1-\frac{\overset{\prime\prime}{a}}{a}\right)\times\left(1-\frac{\overset{\prime\prime}{b}}{b}\right)\times\left(1-\frac{\overset{\prime\prime}{c}}{c}\right)\right]\times\left(1-\frac{\overset{\prime\prime}{p}\overset{\prime\prime}{q}\overset{\prime\prime}{r}}{pqr}\right)$, multiplied by $(1+\varrho)^{-2}$, and that $\left[1-\left(1-\frac{\overset{\prime\prime\prime}{a}}{a}\right)\times\left(1-\frac{\overset{\prime\prime\prime}{b}}{b}\right)\times\left(1-\frac{\overset{\prime\prime\prime}{c}}{c}\right)\right]\times\left(1-\frac{\overset{\prime\prime\prime}{p}\overset{\prime\prime\prime}{q}\overset{\prime\prime\prime}{r}}{pqr}\right)$ multiplied by $(1+\varrho)^{-3}$, will give the present value of the second and third year's rents respectively; and so on for all the subsequent years, to the utmost extent of human life: the sum of all which terms will be the total present value of the annuity required. But these annual expectations being reduced to their simplest terms, and arranged under each other as in the preceding problems, will form fourteen collateral series, the sum of all which will be found equal to $A+B+C-AB-AC-BC+ABC-PQRA-PQRB-PQRC+PQRAB+PQRAC+PQRBC-PQRABC$.

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* Simpson, Prob. 7.

§ 74. Whence it follows that the value of a reversionary annuity, on the longest of any number of lives after any number of joint lives, is equal to the value of an annuity on the longest of all the lives in reversion; *minus* the sum of the values of an annuity on all the joint lives arising from the combination of all the joint lives in possession with each ONE of the other lives; *plus* the sum of the values of an annuity on all the joint lives arising from the combination of all the joint lives in possession with each TWO of the other lives; *minus* the sum of the values of an annuity on all the joint lives arising from the combination of all the joint lives in possession with each THREE of the other lives; and so on.

PROBLEM VII.*

§ 75. To find the value of an annuity on the longest of any number of lives A, B, C, &c, after the extinction of the longest of any number of other lives P, Q, R, &c.

SOLUTION.

From what has been already said in the preceding problems, it will be evident that the annuity, here alluded to, is to continue during the longest of all the given lives A, B, C, P, Q, R; and such would be the value of it, were the lives A, B, C, to come into possession immediately. But as they are to receive nothing during the existence of any one of the lives P, Q, R; the value of an annuity on the longest of their lives must consequently be subtracted.

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* Simpson, Prob. 4.

Whence it follows that, if we subtract the value of an annuity on the longest of the lives in possession from the value of an annuity on the longest of all the lives concerned, the remainder will be the value of the reversionary annuity required.

SCHOLIUM.

§ 76. By the help of these four problems may all the various cases in reversionary annuities be solved. They have been stated at length, in order to give a general view of the subject: but it will readily appear that the combinations of lives, thence arising, are much more numerous than occur in any practical cases; and a ready solution may therefore not immediately present itself amidst the multiplicity of symbols. It rarely happens that more than three lives are involved in any questions of this kind: and in order to prevent any difficulty or confusion in referring to the problems for a solution of such questions, I have selected all the possible cases in which not more than three lives are concerned; to which I have added the algebraic solution of the same: where A, B, P, Q, denote the same as in the preceding problems. These cases are five in number:* viz. to find the value of an annuity

1° On a single life A after another single life P: in which case the value of the reversionary annuity is equal to $A-AP$.

2° On a single life A after the longest of two lives P, Q, in which case the value of the reversionary annuity is equal to $A-AP-AQ+APQ$.

3° On the longest of two lives A, B, after a single life P: in which case the value of the reversionary annuity is equal to $A+B-AB-AP-BP+ABP$.

4° On a single life A after two joint lives PQ: in which case the value of the reversionary annuity is equal to $A-APQ$.

5° On two joint lives AB after a single life P: in which case the value of the reversionary annuity is equal to $AB-ABP$.

For examples of these several cases, see Questions XIII to XVII in Chapter XII.

COROLLARY I.

§ 77. If the contingency of receiving the annuity is not to continue during the whole period of the given lives, but is either *Deferred*† or *Temporary*, we must revert to what has been said in Prob. I, cor. 3 and 4, and substitute the values of such deferred and temporary annuities, deduced from the rules there given, instead of the values of annuities on the whole continuance of those lives: whence by proceeding with these substituted values according to the rules above stated, we shall obtain the required value of the reversionary annuity accordingly.

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* Simpson, Prob. 13 to 17; and his *Sup.* Prob. 15 to 19; Dodson, vol. ii. Ques. 94, 95, 97, 99, and 101; and vol. iii. Ques. 20 27. 28 and 29. Also De Moivre, Prob. 7 and 8, for the first and second cases.

† Price, Note (D).

But this rule may be rendered more convenient for practice by adopting the principles laid down in Prob. II, cor. 2 and 4. Thus, in the first case given in the Scholium, if the contingency is *deferred* for n years, the value of the reversionary annuity will be $(A)^{\mathrm{d}} - (AP)^{\mathrm{d}} = A' \times \frac{\alpha}{a}(1+\varrho)^{-n} - A'P' \times \frac{\alpha\pi}{ap}(1+\varrho)^{-n}$.* And this value subtracted from $A - AP$ will give the value of a similar *temporary* reversionary annuity. The same method of solution will apply to the remaining cases given in the Scholium.

For examples of the use and application of this corollary, see Questions XVIII and XIX in Chapter XII.

COROLLARY II.†

§ 78. If a reversionary annuity, the contingency of enjoying which is deferred for any number of years, depends on the *joint* continuance of all the lives to the end of that term; the value of it will be equal to the value of a reversionary annuity depending on the same number of lives each older by the given term than the given lives, multiplied by the probability that *all* the joint lives shall continue so long, and also by the present value of £1 due at the end of that term. And this case should be carefully distinguished from the deferred reversionary annuity mentioned in the preceding corollary.

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* From the method of notation adopted throughout the present work it will readily appear that π denotes in this case the number of persons living at an age n years older than P.

† Price, Note (C).

For examples of the use and application of this corollary, see the Scholium to Question XIX in Chapter XII.

CHAPTER IV.

On Survivorships.

§ 79. In the preceding chapters I have considered the value of annuities as depending on the continuance of any number of lives, out of any number of given lives; and also the value of reversionary annuities on any number of lives, after the extinction of any number of other lives. I come now to questions of a mixed nature, where the value of the annuity not only depends on the continuance of the given lives, but also on any survivorship between them. In cases of this kind the annuity is frequently enjoyed in different proportions by the persons on whose lives the same is granted; and therefore they are capable of great variety. The nature and extent of such questions will best appear from the following problems.

PROBLEM VIII.*

§ 80. A, B and C agree amongst themselves to purchase an annuity on the longest of their lives, which is to be equally divided between them whilst they are all living; but on the decease of either of them it is to be equally divided between the two survivors, during their joint lives; and then to belong entirely to the last survivor for his life: To find the value of their respective shares, or the proportion which each person ought to contribute towards the purchase.

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* Simpson, Prob. 20; also his *Sup.* Prob. 23. De Meivre, Prob. 9. Dodson, vol. iii. Ques. 73.

SOLUTION.

Let the probabilities of the given lives continuing 1, 2, 3, &c. years be severally denoted by the same characters as in § 23: and let us first determine the share of A. Now the expectation of A, on what he may happen to receive at the end of any one year, may be considered in four parts, as depending on so many different events: 1° A, B and C may be all living, the probability of which at the end of the first year is $\frac{a'b'c'}{abc}$, in which case he will receive $\frac{1}{3}$ of the annuity, or $\frac{1}{3}(1+\varrho)^{-1}$; therefore $(1+\varrho)^{-1} \times \frac{a'b'c'}{3abc}$ will be the value of this expectation: 2° A and B may be living and C dead, the probability of which at the end of the first year is $\frac{a'b'}{ab} \times \left(1 - \frac{c'}{c}\right)$, in which case he will receive $\frac{1}{2}$ the annuity, or $\frac{1}{2}(1+\varrho)^{-1}$; therefore $(1+\varrho)^{-1} \times \frac{a'b'}{2ab} \times \left(1 - \frac{c'}{c}\right)$ will be the value of this expectation: 3° A and C may be living and

B dead, the probability of which at the end of the first year is $\frac{\overset{\prime}{a}\overset{\prime}{c}}{ac}\times\left(1-\frac{\overset{\prime}{b}}{b}\right)$, in which case he will likewise receive ½ the annuity, or $\frac{1}{2}(1+\varrho)^{-1}$; therefore $(1+\varrho)^{-1}\times\frac{\overset{\prime}{a}\overset{\prime}{c}}{2ac}\times\left(1-\frac{\overset{\prime}{b}}{b}\right)$ will be the value of this expectation: 4° A may be the only person living, the probability of which at the end of the first year is $\frac{\overset{\prime}{a}}{a}\times\left(1-\frac{\overset{\prime}{b}}{b}\right)\times\left(1-\frac{\overset{\prime}{c}}{c}\right)$, in which case he comes in for the whole annuity, therefore $(1+\varrho)^{-1}\times\frac{\overset{\prime}{a}}{a}\times\left(1-\frac{\overset{\prime}{b}}{b}\right)\times\left(1-\frac{\overset{\prime}{c}}{c}\right)$ will be the value of this expectation. And the sum of all these values, or $(1+\varrho)^{-1}\times\left(\frac{\overset{\prime}{a}}{a}-\frac{\overset{\prime}{a}\overset{\prime}{b}}{2ab}-\frac{\overset{\prime}{a}\overset{\prime}{c}}{2ac}+\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{3abc}\right)$ will be the total value of the expectation of A, on what he may happen to receive at the end of the first year.

By pursuing the same steps it will be found that $(1+\varrho)^{-2}\times\left(\frac{\overset{\prime\prime}{a}}{a}-\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{2ab}-\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{c}}{2ac}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{3abc}\right)$ will denote the value of his expectation on what he may happen to receive at the end of the second year: and that $(1+\varrho)^{-3}\times\left(\frac{\overset{\prime\prime\prime}{a}}{a}-\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{2ab}-\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{c}}{2ac}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{3abc}\right)$ will denote the value of his expectation on what he may happen to receive at the end of the third year: and so on to the utmost extent of human life. The sum of all which terms, or the series

$$
\begin{aligned}
&(1+\varrho)^{-1}\times\left[\frac{\overset{\prime}{a}}{a}-\frac{\overset{\prime}{a}\overset{\prime}{b}}{2ab}-\frac{\overset{\prime}{a}\overset{\prime}{c}}{2ac}+\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{3abc}\right]+\\
&(1+\varrho)^{-2}\times\left[\frac{\overset{\prime\prime}{a}}{a}-\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{2ab}-\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{c}}{2ac}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{3abc}\right]+\\
&(1+\varrho)^{-3}\times\left[\frac{\overset{\prime\prime\prime}{a}}{a}-\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{2ab}-\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{c}}{2ac}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{3abc}\right]+\\
&\qquad\qquad\&c\qquad\qquad\qquad\&c
\end{aligned}
$$

will be the total value of his interest in the annuity, or the share which he ought to contribute towards the purchase. But the sum of these several perpendicular series is equal to $A-\frac{1}{2}AB-\frac{1}{2}AC+\frac{1}{3}ABC$.

§ 81. As to the share of B, or C, it is evident that their expectation of receiving the annuity in any one year, will depend on the same events, *mutatis mutandis*, as that of A: whence it follows that by substituting the values, thence arising, in the general expression above given, we shall have $B-\frac{1}{2}AB-\frac{1}{2}BC+\frac{1}{3}ABC$ for the value of B's share; and $C-\frac{1}{2}AC-\frac{1}{2}BC+\frac{1}{3}ABC$ for the value of C's share in the given annuity: whence the following rule for either case.

§ 82. *From the value of an annuity on the given life subtract half the sum of the values of an annuity on both the joint lives arising from the combination of the given life with each one of the other; and to the remainder add one third of the value of an annuity on the three joint lives: the sum will be the answer required.*

§ 83. *Example.* Suppose the three lives to be aged 20, 30, and 40; the rate of interest to be 4 per cent, and the probabilities of living as at *Northampton.* Then by referring to the Tables at the end of this work it will be found that the value of the share of each person in this annuity will be as under: viz. that of

$$A = 16{\cdot}033 - \tfrac{1}{2}(11{\cdot}873 + 10{\cdot}924) + \tfrac{1}{3} \times 8{\cdot}986 = 7{\cdot}630$$

$$B = 14{\cdot}781 - \tfrac{1}{2}(11{\cdot}873 + 10{\cdot}490) + \tfrac{1}{3} \times 8{\cdot}986 = 6{\cdot}595$$

$$C = 13{\cdot}197 - \tfrac{1}{2}(10{\cdot}924 + 10{\cdot}490) + \tfrac{1}{3} \times 8{\cdot}986 = \underline{5{\cdot}485}$$

and the sum of all these respective shares, or 19·710 is the value of the annuity on the longest of the three lives; or the total value of the purchase.

COROLLARY I.*

§ 84. If only two lives, A and B, are concerned in the purchase, (during whose joint continuance the annuity is to be enjoyed equally between them, but on the decease of either of them it is to belong wholly to the survivor) the value of the share of A will be $A - \frac{1}{2}AB$; and that of B will be $B - \frac{1}{2}AB$: whence the following rule for two lives.

§ 85. *From the value of an annuity on the given life, subtract half the value of an annuity on the two joint lives: the remainder will be the share of the given life required.*

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* Simpson, Prob. 18, and his *Sup.* Prob. 21. Dodson, vol. iii. Ques. 72.

For examples of the use and application of this corollary, see Question XX in Chapter XII.

COROLLARY II.

§ 86. On the other hand, let the number of lives concerned in the purchase be ever so great, the share of any given life may be readily determined, provided the annuity be always equally divided among the surviving lives. For, let G denote the value of an annuity on the given life; $\overset{\text{I}}{G}$ the sum of the values of an annuity on all the joint lives arising from combining the given life with each *one* of the others; $\overset{\text{II}}{G}$ the sum of the values of an annuity on all the joint lives arising from combining the given life with each *two* of the others; $\overset{\text{III}}{G}$ the sum of the values of an annuity on all the joint lives arising from combining the given life with each *three* of the others; and so on. Then will $G-\frac{1}{2}\overset{\text{I}}{G}+\frac{1}{3}\overset{\text{II}}{G}-\frac{1}{4}\overset{\text{III}}{G}+$ &c denote the share of the given life, or the value which he ought to contribute towards the purchase.

PROBLEM IX.[*]

§ 87. A, B and C agree to purchase an annuity on the longest of their lives, which is to be divided amongst them in the following manner: A and B are to enjoy it equally during their joint lives; but on the decease of either of them it is to be equally divided between A and C, or B and C, the two remaining persons; and lastly, to be enjoyed wholly by the survivor: To find the value of their respective shares.

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[*] Dodson, vol. iii. Ques. 75.

SOLUTION.

The expectation of A on what he may happen to receive at the end of any one year, may be considered in three parts, as depending on so many different events: 1° A and B may be both alive, the probability of which at the end of the first year is $\frac{\acute{a}\acute{b}}{ab}$, in which case he will receive $\frac{1}{2}$ the annuity: 2° A and C may be living and B dead, the probability of which at the end of the first year is $\frac{\acute{a}\acute{c}}{ac} \times \left(1 - \frac{\acute{b}}{b}\right)$ in which case he will also receive $\frac{1}{2}$ the annuity: 3° A may be the only person living, the probability of which at the end of the first year is $\frac{\acute{a}}{a} \times \left(1 - \frac{\acute{b}}{b}\right) \times \left(1 - \frac{\acute{c}}{c}\right)$, in which case he will come in for the whole annuity. Now, the sum of all these values, being multiplied by $(1+\varrho)^{-1}$ will give $(1+\varrho)^{-1} \times \left(\frac{\acute{a}}{a} - \frac{\acute{a}\acute{b}}{2ab} - \frac{\acute{a}\acute{c}}{2ac} + \frac{\acute{a}\acute{b}\acute{c}}{2abc}\right)$ for the total value of the expectation of A on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation on what he may happen to receive at the end of the second, third, and every subsequent year, to the utmost extent of human life. The sum of all which terms, or the series

$$
\begin{array}{l}
(1+\varrho)^{-1}\times\left[\frac{\overset{\prime}{a}}{a}-\frac{\overset{\prime}{a}\,\overset{\prime}{b}}{2ab}-\frac{\overset{\prime}{a}\,\overset{\prime}{c}}{2ac}+\frac{\overset{\prime}{a}\,\overset{\prime}{b}\,\overset{\prime}{c}}{2abc}\right]+\\
(1+\varrho)^{-2}\times\left[\frac{\overset{\prime\prime}{a}}{a}-\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{b}}{2ab}-\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{c}}{2ac}+\frac{\overset{\prime\prime}{a}\,\overset{\prime\prime}{b}\,\overset{\prime\prime}{c}}{2abc}\right]+\\
(1+\varrho)^{-3}\times\left[\frac{\overset{\prime\prime\prime}{a}}{a}-\frac{\overset{\prime\prime\prime}{a}\,\overset{\prime\prime\prime}{b}}{2ab}-\frac{\overset{\prime\prime\prime}{a}\,\overset{\prime\prime\prime}{c}}{2ac}+\frac{\overset{\prime\prime\prime}{a}\,\overset{\prime\prime\prime}{b}\,\overset{\prime\prime\prime}{c}}{2abc}\right]+\\
\qquad\qquad\qquad\quad \&c \qquad\qquad\qquad \&c,
\end{array}
$$

will denote the total value of his interest in the annuity. But, the sum of these several perpendicular series is evidently equal to $A-\frac{1}{2}AB-\frac{1}{2}AC+\frac{1}{2}ABC$: whence we have the following rule.

§ 88. *From the value of an annuity on the life* A, *subtract half the sum of the values of an annuity on the joint lives* AB, *and* AC; *to the remainder add half the value of an annuity on the three joint lives* ABC: *the sum will be the share which* A *ought to contribute towards the purchase.*

§ 89. With respect to the value of B's interest, it is evident that his expectation of receiving the annuity in any one year will depend on the same events, *mutatis mutandis*, as that of A: wherefore by substituting the values, thence arising, in the general expression above given, we shall have $B-\frac{1}{2}AB-\frac{1}{2}BC+\frac{1}{2}ABC$ for the value of B's interest: whence the following rule.

§ 90. *From the value of an annuity on the life* B *subtract half the sum of the values of an annuity on the joint lives* AB, *and* BC; *to the remainder add half the value of an annuity on the three joint lives* ABC; *the sum will be the share which* B *ought to contribute towards the purchase.*

§ 91. But with respect to C's interest, it will appear that his expectation of receiving the annuity in any one year may be considered in the three following points, as depending on so many different events: 1° A and C may be living and B dead, the probability of which at the end of the first year is $\frac{\acute{a}\acute{c}}{ac}\times\left(1-\frac{\acute{b}}{b}\right)$, in which case he will receive $\frac{1}{2}$ the annuity: 2° B and C may be living and A dead, the probability of which at the end of the first year is $\frac{\acute{b}\acute{c}}{bc}\times\left(1-\frac{\acute{a}}{a}\right)$, in which case he will receive also $\frac{1}{2}$ the annuity: 3° C may be the only person living, the probability of which at the end of the first year is $\frac{\acute{c}}{c}\times\left(1-\frac{\acute{a}}{a}\right)\times\left(1-\frac{\acute{b}}{b}\right)$, in which case he will receive the whole annuity. The sum of these values therefore, being multiplied by $(1+\varrho)^{-1}$, will give $(1+\varrho)^{-1}\times\left(\frac{\acute{c}}{c}-\frac{\acute{a}\acute{c}}{2ac}-\frac{\acute{b}\acute{c}}{2bc}\right)$ for the total value of the expectation of C on what he may happen to receive at the end of the first year.

By pursuing the same steps, we may find the value of his expectation on what he may happen to receive at the end of the second, third and every subsequent year to the utmost extent of human life. The sum of all which terms will be the total value of his interest in the annuity; and will be evidently equal to $C-\frac{1}{2}AC-\frac{1}{2}BC$: whence the following rule.

§ 92. *From the value of an annuity on the life* C, *subtract half the sum of the values of an annuity on the joint lives* AC, *and* BC; *the remainder will be the share which* C *ought to contribute towards the purchase.*

§ 93. *Example.* Let the three lives be aged 20, 30, and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton.* The value of the share of each person in this annuity will be as under: viz. that of

$$A=16{\cdot}033-\frac{1}{2}(11{\cdot}873+10{\cdot}924)+\frac{1}{2}\times 8{\cdot}986=9{\cdot}127$$

$$B=14{\cdot}781-\frac{1}{2}(11{\cdot}873+10{\cdot}490)+\frac{1}{2}\times 8{\cdot}986=8{\cdot}093$$

$$C=13{\cdot}197-\frac{1}{2}(10{\cdot}924+10{\cdot}490)\;\ldots\ldots\;=2{\cdot}490$$

And the sum of all these values is the value of an annuity on the longest of the three lives.

PROBLEM X.*

§ 94. A, B and C agree to purchase an annuity on the longest of their lives, to be divided amongst them in the following manner: A and B are to enjoy it equally during their joint lives; if A dies first, then B and C are to enjoy it equally during their joint lives, and the survivor of them is to have the whole; but if B dies first, then A is to enjoy the whole during his life, and after his decease it is to devolve wholly to C: To find the value of their respective shares.

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* Simpson, Prob. 19. Dodson, vol. iii. Ques. 74.

SOLUTION.

The expectation of A, on what he may happen to receive at the end of any one year, may be considered in two points, as depending on so many different events: 1° A and B may be both alive, the probability of which at the end of the first year is $\frac{\acute{a}\acute{b}}{ab}$, in which case he will receive ½ the annuity: 2° A may be living and B dead, the probability of which at the end of the first year is $\frac{\acute{a}}{a} \times \left(1 - \frac{\acute{b}}{b}\right)$, in which case he will enjoy the whole annuity. The sum of these two values being multiplied by $(1+\varrho)^{-1}$ will give $(1+\varrho)^{-1} \times \left(\frac{\acute{a}}{a} - \frac{\acute{a}\acute{b}}{2ab}\right)$ for the total value of the expectation of A on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation on what he may happen to receive at the end of the second, third, and every subsequent year, to the utmost extent of human life. The sum of all which terms will be the total value of his interest in the annuity; and will be evidently equal to $A-\frac{1}{2}AB$: whence we have the following rule.

§ 95. *From the value of an annuity on the life* A, *subtract half the value of an annuity on the joint lives* AB: *the remainder will be the value which* A *ought to contribute towards the purchase.*

§ 96. In order to determine the share of B, it should be observed that his expectation on what he may happen to receive at the end of any one year may be considered in three points, as depending on so many different events: 1° A and B may be both alive, the probability of which at the end of the first year is $\frac{a'b'}{ab}$, in which case he will receive $\frac{1}{2}$ the annuity: 2° B and C may be alive and A dead, the probability of which at the end of the first year is $\frac{b'c'}{bc}\times\left(1-\frac{a'}{a}\right)$, in which case also he will receive $\frac{1}{2}$ the annuity: 3° B may be the only person living, the probability of which at the end of the first year is $\frac{b'}{b}\times\left(1-\frac{a'}{a}\right)\times\left(1-\frac{c'}{c}\right)$, in which case he will enjoy the whole annuity. And the sum of all these values, being multiplied by $(1+\varrho)^{-1}$, will give $(1+\varrho)^{-1}\left(\frac{b'}{b}-\frac{a'b'}{2ab}-\right.$

$\frac{b'c'}{2bc}+\frac{a'b'c'}{2abc})$ for the total value of the expectation of B, on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation for the second, third and every subsequent year, to the utmost extent of human life. The sum of all which terms will be the total value of his interest in the annuity; and will be evidently equal to $B-\frac{1}{2}AB-\frac{1}{2}BC+\frac{1}{2}ABC$: whence the following rule.

§ 97. *From the value of an annuity on the life* B, *subtract half the sum of the values of an annuity on the joint lives* AB, *and* BC; *to the remainder add half the value of an annuity on the three joint lives* ABC: *the sum will be the value which* B *ought to contribute towards the purchase.**

§ 98. It remains now to determine the value of the share of C, whose expectation of receiving the annuity in any one year may be considered in two points: 1° B and C may be both alive and A dead, the probability of which at the end of the first year is $\frac{b'c'}{bc}\times\left(1-\frac{a'}{a}\right)$, in which case he will receive $\frac{1}{2}$ the annuity: 2° C only may be living, the probability of which at the end of the first year is $\frac{c'}{c}\times\left(1-\frac{a'}{a}\right)\times$

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* The value of B's interest in this annuity is the same as that in the preceding Problem.

$(1-\frac{\overset{\prime}{b}}{b})$, in which case he will enjoy the whole annuity. And the sum of these two values, being multiplied by $(1+\varrho)^{-1}$, will give $(1+\varrho)^{-1}\times(\frac{\overset{\prime}{c}}{c}-\frac{\overset{\prime}{a}\overset{\prime}{c}}{ac}-\frac{\overset{\prime}{b}\overset{\prime}{c}}{2bc}+\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{2abc})$ for the total value of the expectation of C, on what he may happen to receive at the end of the first year.

By a similar method of proceeding we may find the value of his expectation for the second, third, and every subsequent year to the utmost extent of human life. The sum of all which terms will express the total value of his interest in the annuity; and will be evidently equal to $C-AC-\frac{1}{2}BC+\frac{1}{2}ABC$: whence the following rule.

§ 99. *From the value of an annuity on the life* C, *subtract the value of an annuity on the joint lives* AC, *and also half the value of an annuity on the joint lives* BC; *to the remainder add half the value of an annuity on the joint lives* ABC: *the sum will be the share which* C *ought to contribute towards the purchase.*

§ 100. *Example.* Suppose the three lives to be aged 20, 30, and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton*. The value of the share of each person in this annuity will be as under: viz. that of

$A = 16{\cdot}033 - \frac{1}{2} \times 11{\cdot}873 \ldots\ldots\ldots\ldots = 10{\cdot}096$

$B = 14{\cdot}781 - \frac{1}{2}(11{\cdot}873 + 10{\cdot}490) + \frac{1}{2} \times 8{\cdot}986 = 8{\cdot}093$

$C = 13{\cdot}197 - 10{\cdot}924 - \frac{1}{2} \times 10{\cdot}490 + \frac{1}{2} \times 8{\cdot}986 = 1{\cdot}521$

and the sum of these three values is the value of an annuity on the longest of the three lives.

PROBLEM XI.*

§ 101. A, B and C agree to purchase an annuity on the longest of their lives, to be divided amongst them in the following manner: A is to enjoy the whole annuity during his life; but after his decease it is to be divided equally between B and C during their joint lives; and the survivor of them is to have the whole: To find the value of their respective shares.

SOLUTION.

The value of A's interest in this annuity, or the share which he ought to contribute towards the purchase, is evidently equal to the value of an annuity on his life: that is, equal to *A*.

§ 102. As to the share of B, the expectation on what he may happen to receive at the end of any one year may be considered in two points, as depending on so many different events: 1° B and C may be alive and A dead, the probability of which at the end

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[*] Simpson's *Sup.* Prob. 22. Dodson, vol. iii. Ques. 76.

of the first year is $\frac{\acute{b}\,\acute{c}}{b\,c}\times\left(1-\frac{\acute{a}}{a}\right)$, in which case he will receive $\frac{1}{2}$ the annuity: 2° B may be the only person living, the probability of which at the end of the first year is $\frac{\acute{b}}{b}\times\left(1-\frac{\acute{a}}{a}\right)\times\left(1-\frac{\acute{c}}{c}\right)$, in which case he will enjoy the whole annuity. And the sum of these two values, being multiplied by $(1+\varrho)^{-1}$, will give $(1+\varrho)^{-1}\times\left(\frac{\acute{b}}{b}-\frac{\acute{a}\,\acute{b}}{ab}-\frac{\acute{b}\,\acute{c}}{2bc}+\frac{\acute{a}\,\acute{b}\,\acute{c}}{2a\,b\,c}\right)$ for the total value of the expectation of B on what he may happen to receive at the end of the first year.

By a similar method of proceeding we may find the value of his expectation for the second, third and every subsequent year, to the utmost extent of human life. The sum of all which terms will be the total value of his interest in the annuity; and will be found equal to $B-AB-\frac{1}{2}BC+\frac{1}{2}ABC$: whence the following rule.

§ 103. *From the value of an annuity on the life* B, *subtract the value of an annuity on the joint lives* AB, *and also half the value of an annuity on the joint lives* BC; *to the remainder add half the value of an annuity on the joint lives* ABC: *the sum will be the value which* B *ought to contribute towards the purchase.*

§ 104. With respect to the share of C, it is evident that his expectation of receiving the annuity at the end of any one year will depend on the same events, *mutatis mutandis*, as that of B. Therefore, by pursuing the same steps, we shall find that the value of his interest in the annuity will be equal to $C-AC-\frac{1}{2}BC+\frac{1}{2}ABC$: whence we have the following rule.

§ 105. *From the value of an annuity on the life* C, *subtract the value of an annuity on the joint lives* AC, *and also half the value of an annuity on the joint lives* BC; *to the remainder add half the value of an annuity on the joint lives* ABC: *the sum will be the value which* C *ought to contribute towards the purchase.**

§ 106. *Example.* Let the three lives be aged 20, 30, and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton*. The value of the share of each person in this annuity will be as under: viz. that of

$A=16{\cdot}033 \ldots\ldots\ldots\ldots\ldots\ldots =16{\cdot}033$
$B=14{\cdot}781-11{\cdot}873-\frac{1}{2}\times10{\cdot}490+\frac{1}{2}\times8{\cdot}986=\ 2{\cdot}156$
$C=13{\cdot}197-10{\cdot}924-\frac{1}{2}\times10{\cdot}490+\frac{1}{2}\times8{\cdot}986=\ 1{\cdot}521$

and the sum of these three values is the value of an annuity on the longest of the three lives.

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* The value of C's interest in this annuity is the same as that in the preceding problem.

PROBLEM XII.

§ 107. A, B and C agree to purchase an annuity on the longest of their lives, to be enjoyed wholly by each of them in succession: that is, A is to enjoy it first, for his life; at his decease, B is to enter upon it next; and C, last: To find the value of their respective shares.

SOLUTION.

The value of A's interest in this annuity, or the share which he ought to contribute towards the purchase, is evidently equal to the value of an annuity on his life: that is, equal to A.

§ 108. The value of B's interest, or the share which he ought to contribute towards the purchase, is equal to the value of a reversionary annuity on his life after the decease of A: that is, (by the Scholium in page 70) equal to $B-AB$.

§ 109. The value of C's interest, or the share which he ought to contribute towards the purchase, is equal to the value of a reversionary annuity on his life after the extinction of the longest of the two lives A and B: that is, (by the same Scholium) equal to $C-AC-BC+ABC$.

§ 110. *Example.* Suppose the three lives to be aged 20, 30, and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton.* The value of the share of each person in this annuity will be as under: viz. that of

$A = 16{\cdot}033 \ldots\ldots\ldots\ldots\ldots = 16{\cdot}033$

$B = 14{\cdot}781 - 11{\cdot}873 \ldots\ldots\ldots\ldots = 2{\cdot}908$

$C = 13{\cdot}197 - (10{\cdot}924 + 10{\cdot}490) + 8{\cdot}986 = {\cdot}769$

and the sum of these three values will be the value of an annuity on the longest of the three lives.

COROLLARY.

§ 111. If only two lives, A and B, are concerned in the purchase, the value of their respective shares will be the same as above given.

PROBLEM XIII.

§ 112. A, B and C agree to purchase an annuity, to continue as long as any two of their lives are in being; and which is to be equally divided between them whilst they are all living; but on the decease of either of them it is to be equally divided between the two survivors during their joint lives: To find the value of their respective shares.

SOLUTION.

The expectation of A, on what he may happen to receive at the end of any one year, may be considered in three parts, as depending on so many different events: 1° A, B and C may be all living, in which case he will receive $\frac{1}{3}$ of the annuity: 2° A and B may be living and C dead, in which case he will receive $\frac{1}{2}$ the annuity: 3° A and C may be living and B dead, in which case he will receive also $\frac{1}{2}$ the annuity. Therefore the sum of these expectations for the first, second, third, &c, years, continued to the utmost extent of human life, will be the total value of A's share in the annuity. But the value of these several expectations has been already found in the solution to Prob. VIII; they being precisely the same as the first three which are there mentioned: and the sum of which, for every year of human life, will be found equal to $\frac{1}{2}AB+\frac{1}{2}AC-\frac{2}{3}ABC$.

§ **113.** With respect to the share of B or C, their expectation of receiving the annuity in any one year will depend on the same events, *mutatis mutandis,* as that of A: whence it follows that by substituting the values, thence arising, in the general expression above given, we shall have $\frac{1}{2}AB+\frac{1}{2}BC-\frac{2}{3}ABC$, for the value of B's interest; and $\frac{1}{2}AC+\frac{1}{2}BC-\frac{2}{3}ABC$, for the value of C's interest in the given annuity: whence the following rule for determining the value of the share of any given life in the annuity.

§ **114.** *Subtract two thirds of the value of an annuity on the three joint lives, from half the sum of the values of an annuity on both the joint lives arising from the combination of the given life with* *each one of the others: the remainder will be the value of the share of such given life.*

§ 115. *Example.* Suppose the three lives to be aged 20, 30, and 40; interest 4 per cent, and the probabilities of living as at *Northampton.* The value of the share of each person will be as under: viz. that of

$$A=\tfrac{1}{2}(11{\cdot}873+10{\cdot}924)-\frac{2}{3}\times 8{\cdot}986=5{\cdot}408$$

$$B=\tfrac{1}{2}(11{\cdot}873+10{\cdot}490)-\frac{2}{3}\times 8{\cdot}986=5{\cdot}191$$

$$C=\tfrac{1}{2}(10{\cdot}924+10{\cdot}490)-\frac{2}{3}\times 8{\cdot}986=4{\cdot}716$$

and the sum of these three values, or 15·315, is the value of an annuity on the three lives, to continue as long as any two of them are in being together.

PROBLEM XIV.

§ 116. A, B and C agree to purchase an annuity, to continue as long as any two of their lives are in being; and which is to be divided amongst them in the following manner: A and B are to enjoy it equally during their joint lives; but on the death of either of them, it is to be equally divided between the two survivors, for their joint lives: To find their respective shares.

SOLUTION.

The expectation of A on what he may happen to receive at the end of any one year may be considered in two parts, as depending on so many different events: 1° A and B may be both alive, in which case he will receive $\frac{1}{2}$ the annuity: 2° A and C may be living and B dead, in which case he will also receive $\frac{1}{2}$ the annuity. Therefore the sum of these expectations for the first, second, third, &c, years, to the utmost extent of human life will be the total value of A's share in the annuity. But the value of these several expectations has been already found in the solution to Prob. IX; they being precisely the same as the first two there mentioned: and the sum of which, for every year of human life, will be found equal to $\frac{1}{2}AB+\frac{1}{2}AC-\frac{1}{2}ABC$: whence the following rule.

§ 117. *Subtract the value of an annuity on the joint lives* ABC, *from the sum of the values of an annuity on each of the joint lives* AB *and* AC: *half the remainder will be the value which* A *ought to contribute towards the purchase.*

§ 118. With respect to the share of B, it is evident that his expectation of receiving the annuity at the end of any one year will depend on the same events, *mutatis mutandis*, as that of A: wherefore by substituting the values, thence arising, in the general expression above given, we shall have $\frac{1}{2}AB+\frac{1}{2}BC-\frac{1}{2}ABC$ for the value of B's interest in the annuity: whence the following rule.

§ 119. *Subtract the value of an annuity on the joint lives* ABC, *from the sum of the values of an annuity on each of the joint lives* AB *and* BC: *half the remainder will be the value which* B *ought to contribute towards the purchase.*

§ 120. But with respect to the share of C, it will appear that his expectation of receiving the annuity at the end of any one year may be considered under the two following points, as depending on so many different events: 1° A and C may be living and B dead, in which case he will receive $\frac{1}{2}$ the annuity: 2° B and C may be living and A dead, in which case also he will receive $\frac{1}{2}$ the annuity. Therefore the sum of these two expectations, for the first, second, third, &c, years to the utmost extent of human life, will be the total value of C's interest in the annuity. But, the value of these expectations has been already found in the solution to Prob. IX; they being precisely the same as the first two which are mentioned in the investigation of C's share in that annuity: and the sum of which, for every year of human life, will be found equal to $\frac{1}{2}AC+\frac{1}{2}BC-ABC$: whence the following rule.

§ 121. *Subtract the value of an annuity on the joint lives* ABC, *from half the sum of the values of an annuity on each of the joint lives* AC *and* BC: *the difference will be the share which* C *ought to contribute towards the purchase.*

§ 122. *Example.* Suppose the three lives to be aged 20, 30, and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton.* The value of the share of each person in this annuity will be as under: viz. that of

$$A = \tfrac{1}{2}(11{\cdot}873 + 10{\cdot}924 - 8{\cdot}986) = 6{\cdot}905$$
$$B = \tfrac{1}{2}(11{\cdot}873 + 10{\cdot}490 - 8{\cdot}986) = 6{\cdot}689$$
$$C = \tfrac{1}{2}(10{\cdot}924 + 10{\cdot}490) - 8{\cdot}986 = 1{\cdot}721$$

and the sum of these three values is the total value of the annuity on the three lives, to continue as long as any two of them are in being together.

PROBLEM XV.

§ 123. A, B and C agree to purchase an annuity to continue as long as any two of them are in being together; and which is to be divided between them in the following manner: A and B are to enjoy it equally during their joint lives; if A dies first, then B and C are to enjoy it equally during their joint lives; but if B dies first, then A is to enjoy the whole during the joint lives of A, and C: To find the value of their respective shares.

SOLUTION.

The expectation of A on what he may happen to receive at the end of any one year may be considered in two parts, as depending on so many different events: 1° A and B may be both alive, the probability of which at the end of the first year is $\frac{a'b'}{ab}$, in which case he will receive $\frac{1}{2}$ the annuity: 2° A and C may be living and B dead, the probability of which at the end of the first year is $\frac{a'c'}{ac} \times \left(1 - \frac{b'}{b}\right)$, in which case he will come in for the whole annuity. The sum of these two values, multiplied by $(1+\varrho)^{-1}$, will give $(1+\varrho)^{-1}$ $\left(\frac{a'b'}{2ab} + \frac{a'c'}{ac} - \frac{a'b'c'}{abc}\right)$ for the total value of the expectation of A on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation for the second, third, and every subsequent year to the utmost extent of human life: the sum of all which terms will be the total value of his interest in the annuity; and will be found equal to $\frac{1}{2}AB + AC - ABC$: whence the following rule.

§ 124. *Add half the value of an annuity on the joint lives* AB, *to the value of an annuity on the joint lives* AC; *from the sum, subtract the value of an annuity on the joint lives* ABC: *the remainder will be the value of the share of* A.

§ 125. The expectation of B on what he may happen to receive at the end of any one year may also be considered in two parts as depending on so many different events: 1° A and B may be both alive, the probability of which at the end of the first year is

$\frac{a'b'}{ab}$, in which case he will receive $\frac{1}{2}$ the annuity: 2° B and C may be alive and A dead, the probability of which at the end of the first year is $\frac{b'c'}{bc}\times\left(1-\frac{a'}{a}\right)$, in which case likewise he will receive $\frac{1}{2}$ the annuity. The sum of these two values therefore being multiplied by $(1+\rho)^{-1}$ will give $(1+\rho)^{-1}\times\left(\frac{a'b'}{2ab}+\frac{b'c'}{2bc}-\frac{a'b'c'}{2abc}\right)$ for the expectation of B on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation on what he may happen to receive at the end of the second, third, and every subsequent year, to the utmost extent of human life: the sum of all which terms will be the total value of his interest in the annuity; and will be found equal to $\frac{1}{2}(AB+BC-ABC)$: whence the following rule.

§ 126. *Subtract the value of an annuity on the three joint lives* ABC, *from the sum of the values of an annuity on each of the joint lives* AB *and* BC: *half the remainder will be the value of the share of* B.*

§ 127. With respect to the share of C, it is solely equal to half the value of a reversionary annuity on the joint lives BC, after the life of A; which reversionary annuity is by the Scholium in page 70, expressed by $\frac{1}{2}(BC - ABC)$: whence the following rule.

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* The value of B's interest in this annuity is the same as that in the preceding problem.

§ 128. *Subtract the value of an annuity on the joint lives* ABC, *from the value of an annuity on the joint lives* BC: *half the remainder will be the value of the share of* C.

§ 129. *Example.* Suppose the three lives to be aged 20, 30, and 40; interest 4 per cent, and the probabilities of living as at *Northampton.* The value of the share of each person will be as under: viz. that of

$$A = \tfrac{1}{2} \times 11{\cdot}873 + 10{\cdot}924 - 8{\cdot}986 = 7{\cdot}874$$
$$B = \tfrac{1}{2}(11{\cdot}873 + 10{\cdot}490 - 8{\cdot}986) = 6{\cdot}689$$
$$C = \tfrac{1}{2}(10{\cdot}490 - 8{\cdot}986) \;.\;.\;.\;. = {\cdot}752$$

and the sum of all these three values will be the total value given for the purchase.

PROBLEM XVI.*

§ 130. D and his heirs, as soon as any *two* of three given lives A,B,C, become extinct, are to enter upon an annuity in order to enjoy the same during the life of the survivor: To find the value of his interest therein.

SOLUTION.

This annuity is evidently to continue during the longest of the three given lives; and the value of the

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[*] Simpson's *Sup.* Prob. 20. Dodson, vol. iii. Ques. 79.

same without any restriction would be (by Prob. II) equal to $A+B+C-AB-AC-BC+ABC$: but since D (or his heirs) is not to receive any thing during the continuance of any two of these lives, the value of an annuity depending on this contingency must be subtracted from the preceding expression; and which value is (by Prob. III) denoted by $AB+AC+BC-2ABC$. Therefore $A+B+C-2AB-2AC-2BC+3ABC$ will be the value required: whence the following rule.

§ 131. *From the sum of the values of an annuity on each of the single lives, subtract twice the sum of the values of an annuity on each pair of joint lives; to the remainder add thrice the value of an annuity on the three joint lives: the sum will be the interest of* D *in this annuity, or the value of the reversion required.*

§ 132. *Example.* An estate is held on three lives whose ages are 20, 30, and 40, the income of which, as soon as any two of these lives become extinct, is to belong to D and his heirs during the continuance of the third life: what is the interest of D in this lease, reckoning the probabilities of living as at *Northampton*, and the rate of interest 4 per cent?

The sum of the values of an annuity on each of the single lives is 16·033 + 14·781 + 13·197 = 44·011; the sum of the values of an annuity on each pair of joint lives is 11·873 + 10·924 + 10·490 = 33·287;

and the value of an annuity on the three joint lives is 8·986. Therefore $44{\cdot}011 - 2 \times 33{\cdot}287 + 3 \times 8{\cdot}986 = 4{\cdot}395$ will be the value required.

PROBLEM XVII.*

§ 133. D and his heirs, as soon as any *one* of three given lives, A, B, C, becomes extinct, are to enter upon an annuity in order to enjoy the same during the continuance of the longest of the remaining lives: To find the value of his interest therein.

SOLUTION.

It is evident in this case also that the annuity is to continue during the longest of the three given lives; and such would be the value of it to D (or his heirs) were he to enter upon it immediately: but since he is not to receive any thing during the continuance of all the joint lives, the value of an annuity on those joint lives must be subtracted from the value of an annuity on the longest of the three lives, in order to obtain the required value. Wherefore, $A+B+C-AB-AC-BC+ABC-ABC=A+B+C-AB-AC-BC$ will be the value of his interest in the annuity: whence the following rule.

§ 134. *From the sum of the values of an annuity on each of the single lives, subtract the sum of the*

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* Dodson, vol. iii, Ques. 78.

values of an annuity on each pair of joint lives: the remainder will be the interest of D *in this annuity, or the value of the reversion required.*

§ 135. *Example.* Suppose the lives to be aged 20, 30 and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton.* In which case the value required would be $16{\cdot}033 + 14{\cdot}781 + 13{\cdot}197 - (11{\cdot}873 + 10{\cdot}924 + 10{\cdot}490) = 10{\cdot}724$.

COROLLARY.

§ 136. Had two lives only, A and B, been concerned in this question, the value of the interest of D in the annuity would (by the same method of reasoning) become equal to $A+B-2AB$: whence the following rule.

§ 137. *From the sum of the values of an annuity on each single life, subtract twice the value of an annuity on the two joint lives: the remainder will be the value required.*

For examples of the use and application of this corollary, see Question XXI in Chapter XII.

PROBLEM XVIII.*

§ 138. D and his heirs, as soon as any *one* of three given lives A, B, C, becomes extinct, are to enter upon an annuity in order to enjoy the same during the joint continuance of the survivors: To find the value of his interest therein.

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* Dodson, vol. iii. Ques. 80.

SOLUTION.

In this case, the annuity is to continue only as long as any two of the given lives are in being; and such would be the value of it to D, were he to enter on it immediately: but since he is to receive nothing during the joint continuance of all the lives, the value of an annuity on those joint lives must be subtracted from the value of an annuity to continue as long as any two of them are in being together. Therefore $AB+AC+BC-2ABC-ABC=AB+AC+BC-3ABC$, will be the value required: whence the following rule.

§ 139. *From the sum of the values of an annuity on each pair of joint lives, subtract thrice the value of an annuity on the three joint lives: the remainder will be the value required.*

§ 140. *Example.* Suppose the lives to be aged 20, 30 and 40; the rate of interest 4 per cent, and the probabilities of living as at *Northampton.* In which case the required value would be $11{\cdot}873+10{\cdot}924+10{\cdot}490-3\times8{\cdot}986=6{\cdot}329$.

SCHOLIUM.

§ 141. The preceding problems contain some of the most general cases of survivorship, where not more than three lives are concerned; they are by no means offered as the whole which may arise, since the conditions of the problems are capable of infinite variety: but, by an attention to the method of deducing the values in the cases here given, it is presumed that no difficulty will be found in investigating any other case which may present itself.

§ 142. I would here observe, however, that the values, in all these problems, are deduced from the *whole continuance* of the lives on which the annuities depend; but if, from the conditions of the problem, the contingencies are *Deferred* or *Temporary*, then the annuities which result therefrom will be affected in a similar manner. In such cases, therefore, we must substitute the value of such deferred or temporary annuities, instead of annuities on the whole continuance of the lives, and proceed with such substituted values in the same manner as with the whole values of the annuities: agreeably to what I have already mentioned in Prob. II, cor. 2 and 4: and cor. 1 to the Scholium in page 70.

CHAPTER V.

On Reversionary Annuities depending upon a particular Order of Survivorship.

§ 143. AMONG the cases of reversionary annuities, involving not more than three lives, and which are given in the Scholium at the end of the third Chapter,* there are two which oftentimes involve a contingency for which it is extremely difficult to find a correct algebraical solution fit for general use. These are the second and fourth cases there mentioned; and the contingency, alluded to, is that one of the two lives *in particular*, P or Q, shall die before or after the other.

As the investigations necessary to the solution of these cases will be of considerable importance in the further progress of this work, I shall devote a separate chapter to the examination of the subject. The solution of the cases themselves will lead to that of some other questions in Survivorships which, for want thereof, could not be inserted in the preceding chapter.

§ 144. Instead of the lives A, P, Q, as given in the scholium above mentioned, let the lives be A, B,

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* See page 71.

C; of which, the lives B and C are supposed to be *in possession*, and the life A *in reversion*. The value of a reversionary annuity on the life A after the joint lives B C, is equal to $A-ABC$: but if A is not to enjoy the annuity unless B be the *first* that dies, of the two joint lives B C, it is evident that the value of the annuity will be considerably affected by this restriction.

In like manner, the value of a reversionary annuity on the life A after the longest of two lives B and C, is equal to $A-AB-AC+ABC$: but if A is not to enjoy the annuity unless B be the *last* that dies, of the two lives BC, it is evident the value of the annuity will in this case also be considerably affected by this restriction.

When the two lives in possession are of the same age, or nearly so, it will be an *equal* chance which of them dies before or after the other: but, as this is not always the case, it would be extremely desirable to obtain an expression, fit for general use, which should express this probability *for every year of human life*; because the expectation of receiving *each* year's rent will depend on such contingency. Unfortunately, however, this probability cannot be denoted by a *constant* quantity till after the extinction of the oldest of the two lives; because the chance of survivorship varies in every succeeding year till that period: and even then we can only approximate to the true value. When the difference between the ages of the lives in possession is not very great, the error will be but trifling (as I have just observed) by supposing it to be an *equal* chance which of those lives dies before or after the other, during the probable time of their joint continuance; but when there is any considerable difference between the ages of the two lives, this cannot be an accurate method of proceeding. *After* that period, however, the contingency may in all cases be expressed with a tolerable degree of correctness. But, previous to the investigation of any of the problems connected with this subject, it will be necessary to premise the following lemmata.

LEMMA I.*

§ 145. To determine the probability that, out of two given lives A and B, one of them in particular, A, shall fail *before* the other.

SOLUTION.

It is evident that this circumstance may take place in any one year, 1° by A dying *in* that year and B surviving it; 2° by the extinction of both the lives *in* the year, A having died first. The probability of the first event happening in the first year is, by § 24 and 25, $\frac{(a-\acute{a})}{a}\times\frac{\acute{b}}{b}$; and the probability of the second event happening in the same period is $\frac{(a-a')}{a} \times \frac{(b-b')}{b} \times \frac{1}{2}$:* the sum of these two, therefore, or $\frac{(a-a') \times (b+b')}{2ab}$, will denote the whole probability that A dies the first of the two lives in the *first* year.

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* Phil. Trans. 1788, part II, page 332. Dodson, vol. iii Ques. 21. Simpson, page 100. De Moivre, Prob. 18.

In like manner, the probability of the first event happening in the second year is $\frac{(a'-a'')b''}{ab}$; and the probability of the second event happening in the same period is $\frac{(a'-a'') \times (b'-b'')}{2ab}$; therefore the sum of these two, or $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\times(\overset{\prime}{b}+\overset{\prime\prime}{b})}{2ab}$, will denote the whole probability that A dies the first of the two lives in the *second* year. And, if this value be added to the one just found for the first year, it will give $\frac{(a-\overset{\prime}{a})\times(b+\overset{\prime}{b})}{2ab}+\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\times(\overset{\prime}{b}+\overset{\prime\prime}{b})}{2ab}$ for the probability that A dies the first of the two lives in *two* years.

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* In all these annual contingencies I have supposed it to be an *equal* chance that A dies before B, whatever may be the difference between the two ages: and though this is not strictly correct, except in those cases where the two lives are equal; yet as such chance is in the present investigation confined to *each particular year*, and, when applied practically, is involved with other quantities which very much diminish the error that would otherwise arise from this hypothesis, it would be needless to render the solution more complicated and intricate by aiming at greater exactness. It will however appear from this remark that the result, here obtained, is only an *approximation* to the true value: which is more and more correct, the further we continue the series.

When A is the youngest of the two lives, the fraction $\frac{1}{2}$ will exceed the chance that A dies before B; and consequently the value of ψ, as found by the lemma, will *exceed* the true value of the probability that A dies before B in any part of their coexistence. On the other hand, when A is the oldest of the two lives, the chance that A dies before B will be more than $\frac{1}{2}$; and consequently ψ will in this case *fall short* of the true value of the probability that A dies before B in such period. Proper allowances may be made in such cases, when there is any material difference in the ages of the two lives.

By a similar method of proceeding it will be found that $\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a})\times(\overset{\prime\prime}{b}+\overset{\prime\prime\prime}{b})}{2ab}$ will denote the whole probability that A dies the first of the two lives in the *third* year: which, being added to the value just found for two years, will give $\frac{(a-\overset{\prime}{a})\times(b+\overset{\prime}{b})}{2ab}+\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\times(\overset{\prime}{b}+\overset{\prime\prime}{b})}{2ab}+\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a})\times(\overset{\prime\prime}{b}+\overset{\prime\prime\prime}{b})}{2ab}$ for the probability that this event will happen in *three* years.

And so on with respect to the values for all the subsequent years to the utmost extent of human life: and, if we make n equal to the difference between the age of the oldest of the given lives and that age in the table of observations at which human life becomes extinct, we shall find that the series $\frac{(a-\overset{\prime}{a})\times(b+\overset{\prime}{b})}{2ab}+\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\times(\overset{\prime}{b}+\overset{\prime\prime}{b})}{2ab}+\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a})\times(\overset{\prime\prime}{b}+\overset{\prime\prime\prime}{b})}{2ab}+\ldots\ldots\frac{(\underset{\prime}{\alpha}-\alpha)\times(\underset{\prime}{\beta}+\beta)}{2ab}$ will express the whole chance of A dying before B, during that period. But this series, reduced to more simple terms, is equal to $\frac{1}{2ab} \times \Big[(a - \overset{\prime}{a}) \times (b + \overset{\prime}{b}) + (\overset{\prime}{a} - \overset{\prime\prime}{a}) \times (\overset{\prime}{b} + \overset{\prime\prime}{b}) + (\overset{\prime\prime}{a} - \overset{\prime\prime\prime}{a}) \times (\overset{\prime\prime}{b} + \overset{\prime\prime\prime}{b}) \ldots\ldots (\underset{\prime}{\alpha} - \alpha) \times (\underset{\prime}{\beta} + \beta) \Big]$, the sum of which I shall denote by ψ; therefore ψ will express the probability that A will die before B during the probable term of their joint continuance: and its application will be shown in the solution of the subsequent problems.

§ 146. In finding the numerical value of this series for lives of different ages, but whose common difference of age is the same, it will be most convenient to begin with the last term of it, or $(\underset{\prime}{\alpha} - \alpha) \times (\underset{\prime}{\beta} + \beta)$; and to proceed upwards till we come to the ages of the given lives, as in finding the numerical value of Annuities: see Prob. I, cor. 2. By this method, the process for finding the probability of survivorship between two other lives, whose ages have the same common difference as the given lives, will be considerably abridged; since the latter terms of the series will be common to both cases.

§ 147. *Example.* Let it be required to find the probability that A the youngest of two lives, whose difference of age is 10 years, shall die before B the eldest; according to the probabilities of living as observed in *Sweden*, amongst males and females *collectively.* The process for finding such values, for every year of human life, will best appear from the following specimen.

Age of A	Age of B	
86	96	$\frac{1}{2\times 144}\times 35 \ldots\ldots = {\cdot}1215$
85	95	$\frac{1}{2\times 2\times 189}\times\left[45\times(2+1)+35\right] \ldots\ldots = {\cdot}2248$
84	94	$\frac{1}{2\times 5\times 244}\times\left[55\times(5+2)+45\times(2+1)+35\right] \ldots\ldots = {\cdot}2275$
83	93	$\frac{1}{2\times 11\times 309}\times\left[65\times(11+5)+55\times(5+2)+45\times(2+1)+35\right] \ldots = {\cdot}2346$
82	92	$\frac{1}{2\times 21\times 384}\times\left[75\times(21+11)+65\times(11+5)+55\times(5+2)+\&c\right] \ldots = {\cdot}2477$
81	91	$\frac{1}{2\times 33\times 468}\times\left[84\times(33+21)+75\times(21+11)+65\times(11+5)+\&c\right] . = {\cdot}2762$
80	90	$\frac{1}{2\times 47\times 558}\times\left[90\times(47+33)+84\times(33+21)+75\times(21+11)+\&c\right] = {\cdot}2999$
&c	&c	&c &c &c &c &c &c

By these examples it will be seen that the probability, that a life aged 80 will die before another life aged 90, is expressed by the fraction ·2999; where *certainty* is denoted by unity: and that, if the two lives were 82 and 92, the probability would be expressed by ·2477. It will be seen also, on inspection, that each year's series is for the most part composed of the terms which form the preceding series: and consequently that the finding the probability of survivorship between lives of *several* ages, whose common difference is the same, is not much more laborious than finding it for the youngest of those lives.

§ 148. The above example relates to the probabilities of survivorship between the two lives as observed in Sweden amongst mankind *at large:* but if one of those lives be a *male* and the other a *female* the results will be materially different, as may be seen by the two following tables, where the answer will vary according as the male or the female is the oldest of the two lives.

Age of A Male.	Age of B Female.	Probability that A dies first.
87	97	·1209
86	96	·1956
85	95	·2405
84	94	·2811
83	93	·3147
82	92	·3342
81	91	·3478
80	90	·3534

Age of A Female.	Age of B Male.	Probability that A dies first.
87	97	·0000
86	96	·0000
85	95	·1228
84	94	·1798
83	93	·2045
82	92	·2437
81	91	·2678
80	90	·2797

Two other tables likewise might be formed for the same ages; viz. for those cases where *both* the lives are males, and for those cases where *both* the lives are females: and the results would in these cases also be different from those above adduced. But enough has been said to enable the reader to calculate the true probability of survivorship according to any case that may come before him.*

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* Mr. Morgan has given a "Table shewing the probabilities of "survivorship between two persons of all ages, whose common "difference of age is not less than 10 years, computed from the "*Northampton* table of observations." See *Phil. Trans.* vol. lxxviii. p. 337: or Dr. Price's *Obs. on Rev. Pay.* vol. i. p. 406. He says that the probabilities are very nearly the same from whatever table of observations they are computed: the *accuracy* of which remark may be best seen by a comparison with the examples above given.

The table, here alluded to, is the only one hitherto calculated for showing the probabilities of survivorship between any two lives. As that table however is applicable only to the valuation of annuities as deduced from the *Northampton* observations, and is moreover adapted to every decade only of human life, I shall point out an easy method whereby we may obtain a *near* value of such probability; which, though not strictly true, will be sufficiently correct if the two lives are between 10 and 70 years of age: at least, till we are possessed of more enlarged tables on this subject. Let the expectation of A's life, as found by Prob. 1, cor. 7, be denoted by e; and let the expectation of B's life as found in the same manner be denoted by ε: then will $\psi = \frac{\varepsilon}{2e}$.

This formula is deduced from the assumption that the decrements of life are equal and uniform at every age, agreeably to the principles of M. De Moivre which will be more fully explained in Chapter IX: and is one of the many instances of the utility and convenience of that celebrated hypothesis. The following ex-

COROLLARY I.

§ 149. If the sum of any number of terms of that series be subtracted from the probability of the failing of the joint lives in that term, the difference will denote the probability that B will die before A in that time. Consequently (since it is certain that one or other of them will die before the end of n years) if the sum of the whole series be subtracted from unity, the difference, or $1 - \psi$, will express the whole chance of B dying before A during the probable term of their joint continuance.

COROLLARY II.

§ 150. When the two lives are equal, the sum of the general series given in the problem is equal to $\frac{1}{2}$: that is, $\psi = \frac{1}{2}$.

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amples, deduced from the *Northampton* observations and compared with the values found by the Lemma, will show how far it may be relied on.

Age of A.	Age of B.	Value by Hypothesis.	Value by Lemma.
20	30	·4228	·4231
20	40	·3452	·3429
20	50	·2692	·2650
20	60	·1976	·1931
30	40	·4082	·4094
30	50	·3182	·3170
30	60	·2330	·2302
40	50	·3897	·3938
40	60	·2862	·2882
50	60	·3671	·3756

SCHOLIUM.

§ 151. Since A and B may denote lives of any ages, and therefore the series given in the problem extends to all cases; whereby ψ becomes a general expression for the probability of survivorship between any two lives: yet, as different probabilities may hereafter arise in the same problem, and as much confusion may be created by the use of the same character to express different quantities, I here take the opportunity of observing that I shall denote the several probabilities of survivorship that may take place between any two of the three lives A, B, C, by the following symbols: viz.

A dying *before* B $= \psi$
A . . . . C $= \varpi$
B . . . . C $= \phi$

whence it follows, by Cor. 1, that the probability of

B dying *before* A $= 1 - \psi$
C . . . . A $= 1 - \varpi$
C . . . . B $= 1 - \phi$

LEMMA II.*

§ 152. To determine the probability that, out of two given lives A and B, one of them in particular, A, shall die *after* the other.

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* Phil. Trans. 1794, part II, page 224. Dodson, vol. iii Ques. 19.

SOLUTION.

It is manifest that this circumstance can take place in the first year only by the extinction of both the lives, A having died last; the probability of which is $\frac{(a-a')}{a} \times \frac{(b-b')}{b} \times \frac{1}{2}$,* an expression which, for reasons that will hereafter appear, I shall make equal to $\frac{a-a'}{a} - \frac{(a-a') \times (b+b')}{2ab}$.

But in the second and following years, the event may take place, 1° by the extinction of both the lives in the year, A having died last; 2° by the decease of A in the year, B having died in either of the preceding years. The probability of the first of these events happening in the second year is $\frac{(a'-a'') \times (b'-b'')}{2ab}$, and the probability of the second event happening in the same period is $\frac{(a'-a'')}{a} \times \left(1 - \frac{b'}{b}\right)$: the sum of these two, therefore, or $\frac{a'-a''}{a} - \frac{(a'-a'') \times (b'+b'')}{2ab}$, will denote the whole probability that A dies after B in the *second* year; and which expression, being added to the value just found for the first year, will give $\frac{a-a'}{a} - \frac{(a-a') \times (b+b')}{2ab} + \frac{a'-a''}{a} - \frac{(a'-a'') \times (b'+b'')}{2ab}$, for the probability that A dies after B in *two* years.

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* See the note in page 109.

In like manner, the probability of the first event happening in the third year is $\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a})\times(\overset{\prime\prime}{b}-\overset{\prime\prime\prime}{b})}{2ab}$, and the probability of the second event happening in the same period is $\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a})}{a}\times\left(1-\frac{\overset{\prime\prime}{b}}{b}\right)$; the sum of which, or $\frac{\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a}}{a}-\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a})\times(\overset{\prime\prime}{b}+\overset{\prime\prime\prime}{b})}{2ab}$, will denote the whole probability of A dying after B in the *third* year: and which expression, being added to the value just found for two years, will give $\frac{a-\overset{\prime}{a}}{a}-\frac{(a-\overset{\prime}{a})\times(b+\overset{\prime}{b})}{2ab}+\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{a}-\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\times(\overset{\prime}{b}+\overset{\prime\prime}{b})}{2ab}+\frac{\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a}}{a}-\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a})\times(\overset{\prime\prime}{b}+\overset{\prime\prime\prime}{b})}{2ab}$, for the probability that A dies after B in *three* years.

And so on with respect to the values for every subsequent year: and if we make n equal to the difference between the age of the oldest of the given lives and that age in the table of observations *at which human life becomes extinct**, we shall find that the

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* Mr. Morgan has taken this term equal only to the difference between the age of the oldest of the given lives and the age of *the oldest life* in the table of observations: whereby he has omitted the chance of one life dying after the other in the *last* year of their joint existence. The reader must bear this in mind when comparing the values, deduced by this rule, with the values deduced from the rule given by Mr. Morgan. I would here observe, that the formula above given is not only the most correct of the two, but also renders the several expressions, in which it is afterwards used, much more simple than those which are deduced according to Mr. Morgan's assumption. And, agreeably to these principles,

series $\frac{a-\overset{\prime}{a}}{a}+\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{a}+\frac{\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a}}{a}+\dots\dots\dots\frac{\underset{\prime}{\alpha}-\alpha}{a}-$ $\left[\frac{(a-\overset{\prime}{a})\times(b+\overset{\prime}{b})}{2ab}+\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\times(\overset{\prime}{b}+\overset{\prime\prime}{b})}{2ab}+\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a})\times(\overset{\prime\prime}{b}+\overset{\prime\prime\prime}{b})}{2ab}+\dots\right.$ $\left.\dots\dots\frac{(\underset{\prime}{\alpha}-\alpha)\times(\underset{\prime}{\beta}+\beta)}{2ab}\right]$ will express the whole chance of A dying after B *during that period*; that is, during the probable term of their joint continuance.

But the first part of this series, or $\frac{a-\overset{\prime}{a}}{a}+\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{a}+\frac{\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a}}{a}+\dots\dots\frac{\underset{\prime}{\alpha}-\alpha}{a}$, is evidently equal to $\frac{a-\alpha}{a}=\left(1-\frac{\alpha}{a}\right)$; that is, equal to the probability that the life A shall fail in that period: and the latter part of the series, which is to be subtracted, is (by the preceding Lemma) equal to ψ: consequently the total required value for n years will be denoted by $\left(1-\frac{\alpha}{a}\right)-\psi$.

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it will be found that the table which Mr. Morgan has given, of the probabilities of one life *dying after* the other, is totally useless: since its application may always be avoided in practice; as will be evident from the method pursued in the investigation of the problems in this, and in the eighth Chapter of the present work.

Should the reader be desirous of knowing the probability that A will die after B during the period of their joint continuance, *according to Mr. Morgan's assumption,* he will find that it is denoted by $\left(1-\frac{\underset{\prime}{\alpha}}{a}\right)-\psi$: that is, by substituting $\underset{\prime}{\alpha}$ for α in the formula which I have given in the text. In like manner the probability of B dying after A in that period will, upon the same assumption, be denoted by by $\psi-\frac{\underset{\prime}{\beta}}{b}$: that is, by substituting $\underset{\prime}{\beta}$ for β in the formula given in the scholium. And so of the other quantities there adduced.

COROLLARY I.

§ 153. If the sum of any number of terms in the above series be subtracted from the probability of both the lives failing in that time, the difference will denote the probability that B will die after A during that period. Consequently if the sum of the whole series above given be subtracted from $\left(1-\frac{\alpha}{a}\right)\times\left(1-\frac{\beta}{b}\right)=1-\frac{\alpha}{a}-\frac{\beta}{b}$,* (or the probability that both the lives fail in that period) the difference, or $\left(1-\frac{\alpha}{a}-\frac{\beta}{b}\right)-\left(1-\frac{\alpha}{a}-\psi\right)=\psi-\frac{\beta}{b}$, will express the chance of B dying after A in that period.

COROLLARY II.

§ 154. When the two lives are both of the same age, ψ becomes equal to $\frac{1}{2}$; as already mentioned in the second corollary to the preceding Lemma: consequently the sum of the series in the present Lemma becomes also equal to $\frac{1}{2}$.

SCHOLIUM.

§ 155. In order to prevent the confusion of quantities alluded to in the Scholium given in page 116, I shall here denote the several probabilities of survivorship, which have been the subject of this Lemma, and which may arise between any two of three lives A, B, C, by the following symbols: viz.

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* Since it is *certain* that one or other of the lives will be extinct at the end of n years, it follows that the quantity $\frac{\alpha\beta}{ab}$, which arises from this product, will vanish altogether.

A dying *after* B $= 1 - \psi - \frac{\alpha}{a}$

A . . . . C $= 1 - \varpi - \frac{\alpha}{a}$

B . . . . C $= 1 - \phi - \frac{\beta}{b}$

whence it follows by Cor. 1, that the probability of

B dying *after* A $= \psi - \frac{\beta}{b}$

C . . . . A $= \varpi - \frac{\gamma}{c}$

C . . . . B $= \phi - \frac{\gamma}{c}$

§ 156. It should here be remarked that the above expressions denote the respective probabilities of survivorship for n years only, or during the probable time of the joint continuance of the two lives; and that the values are deduced without any regard to seniority. Therefore, when A is the oldest of the two lives the general expression in the Lemma will become $1 - \psi$; because α becomes equal to 0, and consequently the fraction $\frac{\alpha}{a}$ vanishes altogether. But, when A is the youngest of the two lives, that expression will not denote the *whole* probability of A dying after B, since there is a further chance of A dying, after having survived B. In order to determine this probability for the subsequent years, the series in the Lemma must be continued till the extinction of A's life: whence it will be found that the probability of A dying after B in $(n+1)$, $(n+2)$, $(n+3)$, &c, years will be respectively denoted by $1-\psi-\frac{\overset{\prime}{\alpha}}{a}$, $1-\psi-\frac{\overset{\prime\prime}{\alpha}}{a}$, $1-\psi-\frac{\overset{\prime\prime\prime}{\alpha}}{a}$, &c ; to the utmost extent of A's life, at which period the expression becomes $1-\psi$.

In like manner, when B is the oldest of the two lives, the probability that B will die after A becomes equal to ψ: but, if B be younger than A, the general expression above given, $\psi-\frac{\beta}{b}$, denotes the probability of that event taking place during n years only, or during the probable time of their joint continuance. And the probability of the same event taking place in $(n+1)$, $(n+2)$, $(n+3)$, &c, years will be respectively denoted by $\psi-\frac{\overset{\prime}{\beta}}{b}$, $\psi-\frac{\overset{\prime\prime}{\beta}}{b}$, $\psi-\frac{\overset{\prime\prime\prime}{\beta}}{b}$, &c ; to the utmost extent of B's life, when the expression at length becomes equal to ψ. The same observations will apply to the other quantities above given.

PROBLEM XIX.*

§ 157. To find the value of a reversionary annuity on the life A, after the longest of two lives B and C, on condition that B dies *after* C.

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* Simpson's *Sup.* Prob. 34. Dodson, vol. iii. Ques. 30. Morgan, Prob. 27, cor. and in Phil. Trans. for 1794, page 240.

SOLUTION.

The chance which A has of receiving this annuity at the end of any one year will depend on the continuance of his life to the end of such term, and on the extinction of both the lives B and C previous thereto; restrained however to the contingency that B dies last. It is this contingency which it is so difficult to represent in such a manner as to be generally useful. In the short space of one year, as I have before observed (page 109), the error is not material by taking one *half* the product of the probabilities that the two lives shall fail in that period. "But, when "the number of years and the difference between the "ages of the two lives are considerable, those chances "must vary in proportion; and therefore unless the "contingency is blended with another which shall "very much diminish the probability of the event, "the solution, by thus indiscriminately supposing the "chances to be equal, must be rendered *extremely* "*inaccurate*."*

§ 158. If the probability of B dying after C in *one*, *two*, *three*, &c, years (as found by the second Lemma) be severally denoted by $\overset{\prime}{y}$, $\overset{\prime\prime}{y}$, $\overset{\prime\prime\prime}{y}$, &c, the expectation which A has of enjoying the annuity at the end of those years will be represented, with a sufficient degree of accuracy,* by $\frac{\overset{\prime}{a}.\overset{\prime}{y}}{a(1+\varrho)}$, $\frac{\overset{\prime\prime}{a}.\overset{\prime\prime}{y}}{a(1+\varrho)^2}$, $\frac{\overset{\prime\prime\prime}{a}.\overset{\prime\prime\prime}{y}}{a(1+\varrho)^3}$, &c, respectively. But, in this case the true value of the reversionary annuity could not be expressed by less than n different series; and therefore would be wholly unfit for general use.

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* These are Mr. Morgan's own words when speaking of Mr. Simpson's method of solution: see *Phil. Trans.* 1791, page 276. We shall find, in the sequel, that he has fallen into the same error himself.

§ 159. It appears, from what has been here said, that the chance of one life dying before or after another differs in *every* year of their joint existence; and that it is not capable of being represented by a *constant* quantity till the extinction of the oldest of such lives. *After* that period, however, the expectation which A has of enjoying the annuity at the end of any subsequent year may be determined sufficiently near for any useful purpose by the help of the preceding lemmata. Consequently, all that appears further necessary for the proper solution of the problem is such an expression as will approximate to the value of the chance that B will die after C during the several years of their *joint* continuance. Let such expression be denoted by χ; (the value of which will be the subject of a future inquiry, see § 173): then will the value of the reversionary annuity, depending on the contingency mentioned in the problem, be determined in the following manner.

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* Mr. Morgan says (*Phil. Trans.* for 1794, p. 238) that these quantities would give the *exact* value of the reversionary annuity: but he asserts this on the presumption that the values deduced from the Lemma give the *true* probabilities of survivorship for *every year* of human life. Whereas those values approximate only in proportion to the length of the series; and are incorrect in the first terms of such series, when there is any considerable inequality between the ages of the two lives. Nevertheless, if they could be at all rendered fit for practical purposes, they would enable us to approximate more nearly to the true value of the reversionary annuity than the inaccurate method hitherto adopted.

§ 160. It is manifest that the payment of the annuity in any one year depends on the continuance of the life A to the end of that year, and on the extinction of both the lives B and C previous thereto, B having died last: the probabilities of which events for the first, second, third, &c, years are respectively denoted by $\frac{\overset{\prime}{a}}{a}\left(1-\frac{\overset{\prime}{b}}{b}\right)\times\left(1-\frac{\overset{\prime}{c}}{c}\right)\chi$, $\frac{\overset{\prime\prime}{a}}{a}\left(1-\frac{\overset{\prime\prime}{b}}{b}\right)\times\left(1-\frac{\overset{\prime\prime}{c}}{c}\right)\chi$, $\frac{\overset{\prime\prime\prime}{a}}{a}\left(1-\frac{\overset{\prime\prime\prime}{b}}{b}\right)\times\left(1-\frac{\overset{\prime\prime\prime}{c}}{c}\right)\chi$, &c. Consequently, the sum of the expectations of receiving the annuity at the end of those periods respectively will, for the first n years,* be denoted by the following series,

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* I would here observe, once for all, that in this and the two subsequent problems I take n to denote the number of years between the age of the oldest life involved in the question, and that age in the table of observations when human life becomes extinct: consequently the value of n will vary according to the three cases given in these problems.

$$(1+\varrho)^{-1}\times\chi\left[\frac{a'}{a}-\frac{a'b'}{ab}-\frac{a'c'}{ac}+\frac{a'b'c'}{abc}\right]+$$
$$(1+\varrho)^{-2}\times\chi\left[\frac{a''}{a}-\frac{a''b''}{ab}-\frac{a''c''}{ac}+\frac{a''b''c''}{abc}\right]+$$
$$(1+\varrho)^{-3}\times\chi\left[\frac{a'''}{a}-\frac{a'''b'''}{ab}-\frac{a'''c'''}{ac}+\frac{a'''b'''c'''}{abc}\right]+$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$(1+\varrho)^{-n}\times\chi\left[\frac{\alpha}{a}-\frac{\alpha\beta}{ab}-\frac{\alpha\gamma}{ac}+\frac{\alpha\beta\gamma}{abc}\right].$$

§ 161. *Case* 1. Let A be the oldest of the three lives. It is evident that in this case the series above given will denote the whole value of the reversionary annuity required: because in the n^{th} year the life A becomes extinct, and all the subsequent terms of the series vanish. But the above series is equal to $\chi[A-AB-AC+ABC]$: that is, equal to the value of a reversionary annuity on the life A after the longest of two lives B and C, multiplied by the chance that B dies after C.*

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* According to the method of solution adopted by Mr. Simpson and Mr. Morgan, the values (in all those cases where A is the oldest life) will be precisely the same, whatever be the difference of age between B and C, whether the contingency depends on B dying after C, or C dying after B. Thus, the value of an annuity on a life aged 78 to be entered upon at the extinction of two lives aged 15 and 75 (which is one of the cases given by Mr. Morgan) is the same whether the contingency depends on the younger life dying after the elder, or on the elder dying after the younger. But it must be evident that an annuity depending on the former contingency is worth more than a similar annuity depending on the latter contingency. For, the time of A's coming into possession is the

§ 162. *Case* 2. Let B be the oldest of the three lives. In this case, the above series will denote the

same in both cases; viz. on the extinction of *both* the lives: therefore the value of the annuity will be affected only by the contingency of one life dying after the other.

Mr. Morgan, in attempting to give a more correct solution for the value of such annuities *after* the extinction of the oldest life, has overlooked the most material part of the process, which is to obtain a more accurate expression for the value of such annuities *during the joint continuance* of all the lives. I am aware that Mr. Morgan asserts that, by taking it as an *equal* chance whether B dies before or after C in any given period, the value of the annuity which results from this assumption will be sufficiently near the *true* value for any useful purpose: and he has given some examples with a view to prove the accuracy of his remark. I shall here, however, take the opportunity of observing that what he calls the *true* value is only an *approximation*, which differs most from the true value in those very cases where it is most wanted as a test. He has deduced certain values from a false hypothesis; and afterwards, assuming these values as if *correct*, endeavours to prove that another method of approximation used by him is accurate because it agrees with these assumed values. Now, from what has been said in the preceding pages, I think it must be evident that, when there is any considerable difference between the ages of the two lives, the value of the probability deduced by the lemmata will not be the correct value for *every year* of human life: neither will the method of proceeding, alluded to in § 158, enable us in such cases to obtain the true value of the reversionary annuity.

These observations apply to the table inserted by Mr. Morgan in the *Phil. Trans.* for 1794, page 234, and to the examples given by him in page 239 of the same volume. Mr. Morgan says "that "the approximations and exact values do not differ much from "each other till the last years of the oldest life:" but the fact is that they differ nearly in the same ratio throughout the whole time of the joint continuance of the two lives. His own remarks prove the inaccuracy of his method of reasoning.

value of the reversionary annuity for the first n years (*n now* denoting the number of years between the age of B and that age in the table of observations when human life becomes extinct): and the sum of it will be found equal to $\chi\left[(A)^{t} - AB - (AC)^{t} + ABC\right]$. For, the first and third of these perpendicular series being continued to n terms only, the sum of those terms will, by Prob. I, cor. 4, be accurately represented by the characters here given: and the second and fourth perpendicular series evidently denote the *whole* value of the annuities on those lives respectively; since in the n^{th} year the life B becomes extinct and all the subsequent terms of those series vanish.

Now, in order to determine the expectation of receiving the several rents in the remaining years of A's life, it should be observed that the payment of the annuity in any one year depends on the continuance of the life A to the end of that year, and on the probability that B dies after C previous thereto: but the probability, that B will die after C in the $(n+1)^{\text{st}}$ and all the subsequent years, is (by the scholium to the second Lemma) denoted by the constant quantity $(1-\phi)$.* Consequently the sum of the expectations of receiving the $(n+1)^{\text{st}}$, $(n+2)^{\text{nd}}$, $(n+3)^{\text{rd}}$, &c, year's rents will be expressed by the series $\frac{(1-\phi)\acute{\alpha}}{a(1+\varrho)^{n+1}} +$

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* Because, in the general expression there given, β becomes equal to 0 when B is the oldest life; and the fraction $\frac{\beta}{b}$ consequently vanishes.

$\frac{(1-\phi)\overset{\prime\prime}{\alpha}}{a(1+\varrho)^{n+2}} + \frac{(1-\phi)\overset{\prime\prime\prime}{\alpha}}{a(1+\varrho)^{n+3}} +$ &c: which, being continued for all the subsequent years of A's life, will show the true value of all the rents to be received after n years. But the sum of this series is, by Prob. I, cor. 3, equal to $(1-\phi)\times(A)^{\mathrm{d}}$; and which, being added to the first n terms of the several collateral series above found, will make the total value of the annuity in this case equal to $\chi\left[(A)^{\mathrm{t}}-AB-(AC)^{\mathrm{t}}+ABC\right]+(1-\phi)\times(A)^{\mathrm{d}}$. But, since $(A)^{\mathrm{t}}=A-(A)^{\mathrm{d}}$, and $(AC)^{\mathrm{t}}=AC-(AC)^{\mathrm{d}}$, as appears from Prob. I, cor. 4, it follows that this value may be more conveniently expressed by $\chi[A-AB-AC+ABC]+(1-\phi-\chi)\times(A)^{\mathrm{d}}+\chi(AC)^{\mathrm{d}}$.*

§ 163. *Case* 3. Let C be the oldest of the three lives. The value of all the rents for the first n years (n *now* denoting the number of years between the age of C and that age in the table of observations when human life becomes extinct) may in this case, as in the last, be denoted by the first n terms of the series given in the Problem; and the sum of which will now be equal to $\chi\left[(A)^{t}-(AB)^{t}-AC+ABC\right]$. For, the first and second of those perpendicular series being continued to n terms only, the sum of those terms will be truly denoted by the characters here given: but the last two perpendicular series evidently denote the value of annuities on the *whole continuance* of those lives respectively; because in the n^{th} year the life C is extinct.

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* I would here observe that the rule given by Mr. Simpson for the solution of this case is expressed by $(A-2AB+ABB)\frac{\varepsilon}{\eta 2}$: where ε denotes the expectation of the life B as deduced from the rule in page 49, and where η denotes the expectation of the life C as deduced from the same rule.

When C is the oldest life, this value must be deducted from the whole value of an annuity on the life A after the longest of the two lives B and C; agreeably to the method which will be explained in the note in page 140.

Now in order to determine the expectation of receiving the several rents in the remaining years of A's life, it should be observed that the probability of B dying after C in $(n+1)$, $(n+2)$, $(n+3)$, &c, years is, by the scholium to the second Lemma, respectively denoted by $\left(1-\phi-\frac{\overset{\prime}{\beta}}{b}\right)$, $\left(1-\phi-\frac{\overset{\prime\prime}{\beta}}{b}\right)$, $\left(1-\phi-\frac{\overset{\prime\prime\prime}{\beta}}{b}\right)$, &c: consequently, since the payment of the annuity in any one year depends on the continuance of A's life to the end of that year and on B having died after C previous thereto, it follows that if these values be severally multiplied by $\frac{\overset{\prime}{a}}{a}$, $\frac{\overset{\prime\prime}{a}}{a}$, $\frac{\overset{\prime\prime\prime}{a}}{a}$, &c, (or the probability that A will live $(n+1)$, $(n+2)$, $(n+3)$, &c, years respectively) the products thence arising will express the whole chance of receiving each year's rent respectively. And these products, being again multiplied by the present value of £1 due at the end of those respective periods, will give the expectations of receiving such sum at the end of those years: and which will be expressed by the following series,

$$\begin{array}{l}
(1+\varrho)^{-(n+1)}\times\left[(1-\phi)\dfrac{\overset{\prime}{\alpha}}{a}-\dfrac{\overset{\prime}{\alpha}\overset{\prime}{\beta}}{ab}\right]+\\
(1+\varrho)^{-(n+2)}\times\left[(1-\phi)\dfrac{\overset{\prime\prime}{\alpha}}{a}-\dfrac{\overset{\prime\prime}{\alpha}\overset{\prime\prime}{\beta}}{ab}\right]+\\
(1+\varrho)^{-(n+3)}\times\left[(1-\phi)\dfrac{\overset{\prime\prime\prime}{\alpha}}{a}-\dfrac{\overset{\prime\prime\prime}{\alpha}\overset{\prime\prime\prime}{\beta}}{ab}\right]+\\
\quad\&c \qquad\qquad\qquad \&c \qquad\quad \&c.
\end{array}$$

But the sum of these two perpendicular series, continued to the utmost extent of A's life, is by Prob. I, cor. 3, equal to $(1-\phi)\times(A)^{\mathrm{d}}-(AB)^{\mathrm{d}}$; and which, being added to the first n terms of the several collateral series above found, will make the total value of the annuity in this case equal to $\chi\left[(A)^{\mathrm{t}}-(AB)^{\mathrm{t}}-AC+ABC\right]+(1-\phi-\chi)\times(A)^{\mathrm{d}}-(1-\chi)\times(AB)^{\mathrm{d}}$: an expression which, from what has been said in the last case, may be more conveniently denoted by $\chi\left[A-AB-AC+ABC\right]+(1-\phi-\chi)\times(A)^{\mathrm{d}}-(1-\chi)\times(AB)^{\mathrm{d}}$.

COROLLARY.

§ 164. If the two lives in possession are both of the same age, B, then ϕ and χ each become equal to $\frac{1}{2}$, and the quantities $(AB)^{\mathrm{d}}$ and $(AC)^{\mathrm{d}}$ vanish altogether: consequently the expression in each case becomes $\frac{1}{2}[A-2AB+ABB]$; that is, equal to half the value of a reversionary annuity on the life A after the longest of two equal lives B and B.

But, if the two lives in possession are of the same age with the life A in reversion, this expression will become $\frac{1}{2}[A-2AA+AAA]$.

PROBLEM XX.*

§ 165. To find the value of a reversionary annuity on the life A, after two joint lives BC, on condition that B dies *before* C.

SOLUTION.

The payment of this annuity at the end of any one year depends on the continuance of the life A to the end of that year, and on the extinction of the life B before C previous thereto. As this latter contingency cannot be conveniently represented with accuracy for every year of human life, it becomes necessary, as in the last problem, to have recourse to an approximation during the number of years that the lives have a chance of continuing together. Therefore let the same symbols be used as in that problem: and let us consider the payment of the annuity, during the first n years, as depending in each year on one or other of two events: 1° that A lives to the end of the year, and that B and C both fail previous thereto B having died first: 2° that only B dies previous thereto, and that A and C both live. The probability of the happening of the first event in the first year is $\frac{\overset{\prime}{a}}{a}\left(1-\frac{\overset{\prime}{b}}{b}\right)\times\left(1-\frac{\overset{\prime}{c}}{c}\right)\times(1-\chi)$;* and the probability of the happening of the second event in the same period is $\left(1-\frac{\overset{\prime}{b}}{b}\right)\times\frac{\overset{\prime}{a}\overset{\prime}{c}}{ac}$: therefore the sum of these two, reduced to simple terms and multiplied by $(1+\varrho)^{-1}$, will give $(1+\varrho)^{-1}\times\left[\frac{\overset{\prime}{a}}{a}-\frac{\overset{\prime}{a}\overset{\prime}{b}}{ab}-\left(\frac{\overset{\prime}{a}}{a}-\frac{\overset{\prime}{a}\overset{\prime}{b}}{ab}-\frac{\overset{\prime}{a}\overset{\prime}{c}}{ac}+\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}\right)\chi\right]$ for the total value of the expectation of receiving the first year's rent. By a similar method of reasoning we may find the expectation of receiving the second, third, &c, year's rents: and the sum of all these expectations for the first n years, will be denoted by the following series,

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* Simpson's *Sup.* Prob. 35. Dodson, vol. iii. Ques. 49. Morgan, Prob. 27: and in Phil. Trans. for 1794, page 235.

$$(1+\varrho)^{-1}\times\left[\frac{\overset{\prime}{a}}{a}-\frac{\overset{\prime}{a}\overset{\prime}{b}}{ab}-\left(\frac{\overset{\prime}{a}}{a}-\frac{\overset{\prime}{a}\overset{\prime}{b}}{ab}-\frac{\overset{\prime}{a}\overset{\prime}{c}}{ac}+\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}\right)\chi\right]+$$

$$(1+\varrho)^{-2}\times\left[\frac{\overset{\prime\prime}{a}}{a}-\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{ab}-\left(\frac{\overset{\prime\prime}{a}}{a}-\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{ab}-\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{c}}{ac}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}\right)\chi\right]+$$

$$(1+\varrho)^{-3}\times\left[\frac{\overset{\prime\prime\prime}{a}}{a}-\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{ab}-\left(\frac{\overset{\prime\prime\prime}{a}}{a}-\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{ab}-\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{c}}{ac}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}\right)\chi\right]+$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$(1+\varrho)^{-n}\times\left[\frac{\alpha}{a}-\frac{\alpha\beta}{ab}-\left(\frac{\alpha}{a}-\frac{\alpha\beta}{ab}-\frac{\alpha\gamma}{ac}+\frac{\alpha\beta\gamma}{abc}\right)\chi\right].$$

§ 166. *Case.* 1. Let A be the oldest of the three lives. It is evident that in this case the series above given will express the whole value of the reversionary annuity required: because in the n^{th} year the life A becomes extinct, and all the subsequent terms of the series vanish. But the above series is in such case evidently equal to $A - AB - \chi[A - AB - AC + ABC]$.

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* Since χ denotes the chance that B will die *after* C in every year of their joint existence, it follows that $1-\chi$ will denote the chance that B will die *before* C in the same periods.

§ 167. *Case* 2. Let B be the oldest of the three lives. In this case, the above series will denote the value of the reversionary annuity for the first n years, and the sum of it will be found to be $(A)^{\mathrm{t}} - AB - \chi\left[(A)^{\mathrm{t}} - AB - (AC)^{\mathrm{t}} + ABC\right]$. In the subsequent years, the chance of receiving the annuity will depend on the continuance of the life A, and on the probability of B dying before C: therefore the value of the annuity for the remaining years will be denoted by the sum of the series $\frac{\phi . \overset{\prime}{\alpha}}{a(1+\varrho)^{n+1}} + \frac{\phi . \overset{\prime\prime}{\alpha}}{a(1+\varrho)^{n+2}} + \frac{\phi . \overset{\prime\prime\prime}{\alpha}}{a(1+\varrho)^{n+3}} + \&c = \phi(A)^{\mathrm{d}}$. Consequently the total value of the required annuity will be $A - AB - \chi[A - AB - AC + ABC] - (1 - \phi - \chi) \times (A)^{\mathrm{d}} - \chi(AC)^{\mathrm{d}}$.

§ 168. *Case* 3. Let C be the oldest of the three lives. In this case, the above series will also denote the value of the reversionary annuity for the first n years; the sum of which will now become equal to $(A)^{\mathrm{t}} - (AB)^{\mathrm{t}} - \chi\left[(A)^{\mathrm{t}} - (AB)^{\mathrm{t}} - AC + ABC\right]$:

and the value of the annuity for the remaining years will be truly expressed, as in the last case, by $\phi\,(A)^{d}$. Consequently the total value of the annuity required will be $A - AB - \chi[A - AB - AC + ABC] - (1 - \phi - \chi) \times (A)^{d} + (1 - \chi) \times (AB)^{d}$.

COROLLARY.

§ 169. If the two lives in possession are both of the same age B, then ϕ and χ will each (as in the corollary to the last problem) become equal to $\frac{1}{2}$, and the quantities $(AB)^{d}$ and $(AC)^{d}$ vanish altogether: consequently the value of the reversionary annuity in each case is expressed by $\frac{1}{2}[A - ABB]$.

But when the two lives in possession are of the same age as the life A in reversion, this expression will become equal to $\frac{1}{2}[A - AAA]$.

SCHOLIUM.

§ 170. If the value found according to either of the three cases in this problem be added to the value found according to the corresponding case in the preceding problem, their sum will be equal to $A - AB$. Consequently, if the value of the annuity in either case of the preceding problem be once determined, the value of the same annuity according to the corresponding case in the present problem may be easily found (provided the ages of the three lives A, B, C, are the same in both instances) by subtracting the former value from $A-AB$: and *vice versâ*.

This is almost self-evident: for, taking the contingencies of both problems, it is manifest that the annuity is certain to be enjoyed by A after the decease of B, provided he lives so long. Consequently the sum of the values found by both problems must in each case be equal to the value of a reversionary annuity on the life A after the decease of B. The truth of this scholium is indeed evident on inspection.

PROBLEM XXI.*

§ 171. B and C possess an annuity between them; which, if B survives C, is afterwards to be equally divided between A and B during their joint existence, and then is to go entirely to the survivor for his life: To find the value of the share of A in this annuity.

SOLUTION.

Since A is not to receive any thing if C survives B, it is evident that he is entitled to half the value of a reversionary annuity on the joint lives AB after the extinction of the life C; and also to the whole value of a reversionary annuity on his own life after the extinction of the life B, provided B dies after C. The former value is, by the scholium in page 70, equal to $\frac{1}{2}(AB-ABC)$; and the latter value may be found by Problem XIX, according as A, B, or C, is the oldest of the three lives: whence, the following values may be readily deduced, for the three cases there mentioned.

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* Simpson, Prob. 29, and *Sup.* Prob. 36.

That is, when A is the oldest of the three lives, the required value will be equal to $\chi(A-AC)+(\frac{1}{2}-\chi)\times(AB-ABC)$.* When B is the oldest of the three lives, it will be equal to $x(A - AC) + (\frac{1}{2} - x) \times (AB - ABC) + (1 - \phi - x) \times (A)^{\text{d}} + x(AC)^{\text{d}}$. When C is the oldest of the three lives, it will be equal to $x(A - AC) \times (\frac{1}{2} - x) \times (AB - ABC) + (1 - \phi - x) \times (A)^{\text{d}} - (1 - x) \times (AB)^{\text{d}}$. When the two lives in possession are of the same age B, it will be equal to $\frac{1}{2}(A - AB)$.* And when all the lives are of the same age A, it will be equal to $\frac{1}{2}(A - AA)$.†

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* If we take χ to denote $\frac{1}{2}$ (that is, if we suppose with Mr. Morgan indiscriminately that it is an even chance whether B dies before or after C in any period of their joint existence, whatever be the difference of age between the two lives) then will this formula become equal to $\frac{1}{2}(A-AC)$: and this is the method of solution adopted also by Mr. Simpson. Mr. Morgan, however, in his hurry to attack M. De Moivre's hypothesis, has inadvertently called this formula an *absurd* one; and says that the error arises from Mr. Simpson's having been misled by that *hypothesis* in determining the probability of one life dying after the other: see *Phil. Trans.* vol. lxxxi, page 276. The present investigation, *from the real probabilities of life,* will show that accusation to be unfounded. Mr. Morgan has wholly mistated the case, as he will readily perceive on re-perusing what he has there written: he has confounded Mr. Simpson's 34th and 36th problems together; and thereby brought an unmerited censure upon that author. The error does not arise from the use of De Moivre's hypothesis, (as he would *wish* us to believe) but from the inaccurate method which he, *in common with Mr Morgan*, has adopted in order to express the chance of one life dying before or after the other during the probable time of their joint continuance: and the same *absurd* formula (if I may retort the self-confuting charge) will equally arise, as I have above observed, even according to Mr. Morgan's own method of solution.

GENERAL SCHOLIUM.

§ 172. It now remains only to determine the value of x in order to obtain the proper solution of these three problems: and, if the chance of one life dying before or after another in every year of their joint existence were in a constant ratio, we should find no difficulty therein. But since (in computing from the real probabilities of life) this chance is continually varying, we must have recourse to an approximation towards the mean value of such ratio.

§ 173. Now I have found, from a number of repeated trials, that the value of χ may, when B is the youngest of the two lives B and C, be safely expressed by $\frac{(1-\phi)b-\beta}{b-\beta}$; and that when C is the youngest of the two lives B and C it may be safely expressed by $\frac{(1-\phi)c}{c-\gamma}$: which, though not in all cases strictly correct,* will come nearer to express the true value of the reversionary annuity than by making χ indiscriminately equal to $\frac{1}{2}$, whatever be the ages of B and C; and may be used till its true value be more correctly determined. But, should a more accurate expression for the value of χ be hereafter found, the general solution of the three problems will not be at all affected thereby, since we may give to χ all possible values.

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* This formula will not differ much from the true value, when the two lives in possession are *nearly* of the same age.

† Many other cases of survivorship might be produced, which involve the contingencies mentioned in Problems XIX and XX; and which are solved by the help of those problems in the manner here stated. But, after this investigation, I presume the reader will not find any difficulty in the solution of any other questions of this kind that may occur in practice.

I shall now insert a few examples, in order to show the use and application of these several formulæ.

§ 174. *Example* 1. What is the value of an annuity on the life of A aged 60 after B aged 40, provided B dies *after* another life C aged 20: interest being reckoned at 4 per cent, and the probabilities of living as at *Northampton?*

Here we shall have $A = 9{\cdot}039$, $AB = 7{\cdot}490$, $AC = 7{\cdot}995$, $ABC = 6{\cdot}722$, and $\chi = {\cdot}395$: consequently the value of the reversionary annuity required will in this case be equal to $\cdot 276 \times \cdot 395 = \cdot 109$. But, if B had been 20 and C 40 years of age, we should have $z = {\cdot}605$: consequently the value of the reversionary annuity would in this case be equal to $\cdot276 \times \cdot605 = \cdot167$.

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* It cannot be strictly correct, because the *true* value of ϕ cannot be deduced by the method pursued in the lemmata; as I have already observed in the note in page 109.

In like manner, if A had been 20, B 60, and C 40 years of age, we should have $A = 16{\cdot}083$, $AB = 7{\cdot}995$, $AC = 10{\cdot}924$, $ABC = 6{\cdot}722$, $n = 37$, $z = {\cdot}354$, and $\phi = {\cdot}7118$: consequently the value of the reversionary annuity would in this case be equal to $3{\cdot}836 \times {\cdot}354 - {\cdot}067 + {\cdot}026 = 1{\cdot}317$. But, if B had been 40 and C 60 years of age, the value would in such case come out equal to 2·519.*

§ 175. *Example* 2. What is the value of an annuity on the life of A aged 60 after B aged 40, provided B dies *before* another life C aged 20: interest at 4 per cent, and the probabilities of living as at *Northampton?*

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* It may here be useful to remark that, if the value of the annuity on the life A after the longest of the two lives B and C (provided B dies after C) be once found, we may readily determine the value of the same annuity, on the contingency that C dies after B, by subtracting the value, thus found, from the whole value of the reversionary annuity on the life A after the longest of the two lives B and C. Thus, the value found by the example given in the text being equal to 1·317, and the latter value here alluded to being equal to 3·836, it follows that $3{\cdot}836 - 1{\cdot}317 = 2{\cdot}519$ will be the value required.

In order that the reader may see the difference in the results, according to Mr. Morgan's formulæ and those which are here given, I shall insert the following comparative values of annuities

Here we shall have $A - AB = 9{\cdot}039 - 7{\cdot}490 = 1{\cdot}549$: consequently $1{\cdot}549 - {\cdot}109 = 1{\cdot}440$ will (agreeably to what has been said in the scholium in page 135) be the value of the reversionary annuity in this case required. But if A had been 20 and C 60 years of age the value of the reversionary would in such case be equal to $5{\cdot}109 - 2{\cdot}519 = 2{\cdot}590$.

on the life A after B, provided B dies after C, deduced from the Northampton tables, and reckoning interest at 4 per cent: which difference arises from the inaccurate method, adopted by Mr. Morgan, of taking it as an *equal* chance in all cases that B will die after C, whatever be their difference of age.

Age of A.	Age of B.	Age of C.	Value by Baily.	Value by Morgan.
60	40	20	·109	·138
60	20	40	·167	·138
20	60	40	1·317	1·717
20	40	60	2·519	2·120
40	60	20	·519	·716
40	20	60	·986	·789

CHAPTER VI.

On Assurances.

§ 176. In the preceding chapters I have considered the present value of sums of money as depending on the existence of any given lives, or on any particular survivorship between them; and, in the solution of the different problems relative thereto, have had regard only to the probability of the *living* of those persons on whom the annuity was considered as depending. I come now, however, to treat of those cases where it is required to find the value of annuities, or of sums of money, depending on the *extinction* of any lives; or, in other words, to treat of the value of *Assurances on lives:** a term applied to that compact whereby security is granted for the payment of an annuity or sum of money on the expiration of the lives on which the grant is made, in consideration of such a previous payment made to the assurer, as is accounted a sufficient compensation for the chance of loss to which he exposes himself. The value of this payment (commonly called the *Premium*), in all the principal cases which arise out of this subject, it is my object in the present chapter to determine.

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* The term *Assurance* is usually applied only to the value of annuities or sums of money to be paid *after the extinction* of any given lives: but, it may, with equal propriety, be applied to the value of those annuities which are paid *during the existence* of any given lives; and which have been the subject of the preceding chapters. For, if I give a sum of money for the grant of an annuity during the continuance of any given lives, I give such sum in order to have the annuity *assured* to me; which, without this warrantry would be precarious and uncertain. As I am ignorant of any other word, but such as would be equally ambiguous and indefinite, I have used the term *assurance*, in its most common acceptation, to express the values treated of in the present chapter.

§ 177. It may here, however, be necessary previously to observe that the method to be pursued, for determining the value of any *sum* depending on the extinction of any given lives, will be materially different from that which is pursued for determining the value of any *annuity* under the same circumstances. In the latter case, the expectant is to receive *several* yearly rents, the expectation of receiving *each* of which is independent of his expectation of receiving any other of them. But, in reversionary *sums* the case is very different: for, here, only *one gross sum* is to be received at the extinction of the given lives; and therefore the expectation of receiving it at the end of any one year will depend on its *not* having been received in any preceding year: or, which is the same thing, the chance of receiving the sum at the end of any year will be compounded of the probability of the given lives failing *in* that year, and of their having continued through all the preceding years. This, however, will more fully appear in the following investigations.

PROBLEM XXII.*

§ 178. To determine the present value of a given sum payable at the end of the year in which any number of lives become extinct.

SOLUTION.

Let us in the present investigation confine the case to three joint lives A B C, whose probabilities of continuing 1, 2, 3, &c, years are as expressed in § 24: and let the given sum be denoted by s. Now, the present value of such sum, certain to be received at the end of one year, is equal to $s(1+\varrho)^{-1}$: but, as the chance of receiving this sum, at the end of the first year, depends on the joint lives failing in that year, the probability of which is $\frac{abc-\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}$, we must multiply the present value of the sum above mentioned by this probability; which will give $s(1+\varrho)^{-1} \times \frac{abc-\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}$ for

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[*] Simpson, Prob. 21, and *Sup.* Prob. 26. De Moivre, Prob. 16. Dodson, vol. ii. Ques. 89. Morgan, Prob. 8. Price, Note (E). On referring, however, to the first three authors, here alluded to, it will be found that their investigations are erroneous; in as much as they consider a given *sum* as equivalent to the perpetuity of an *annuity* (commencing immediately) equal to the interest of such sum. See the Scholium in § 107.

the true value of the first year's expectation, or of the chance of receiving such sum at the end of the first year.

In like manner, the present value of the given sum, certain to be received at the end of two years, is equal to $s(1+\varrho)^{-2}$: but, as the chance of receiving such sum at the end of the second year depends on the joint lives failing *in* that year, the probability of which (by § 27) is $\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}-\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}$, we must multiply the present value of the sum above mentioned by this probability; which will give $s(1+\varrho)^{-2}\times\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}-\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}$ for the true value of the expectation of receiving such sum at the end of the second year.

And by a similar method of proceeding it will be found that, since the chance of receiving the given sum at the end of the third year depends on the joint lives failing *in* that year, the probability of which is $\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}-\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}$, we shall have $s(1+\varrho)^{-3}\times\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}-\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}$ for the true value of the expectation of receiving such sum at the end of the third year. And so on for every subsequent year of human life: the sum of all which values will be the required present value of the given sum.

§ 179. But the sum of these quantities, reduced to their simplest terms, is equal to the two following series; $\frac{s}{abc(1+\varrho)}\left[abc+\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{(1+\varrho)}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{(1+\varrho)^3}+\&c\right]$

$-\frac{s}{abc}\left[\frac{a'b'c'}{(1+\varrho)}+\frac{a''b''c''}{(1+\varrho)^2}+\frac{a'''b'''c'''}{(1+\varrho)^3}+\&c\right]$: the former of which is equal to $\frac{s}{(1+\varrho)}(1+ABC)$; and the latter, which is to be subtracted, is equal to $s\times ABC$. Consequently the total present value required becomes $s\left[\frac{1+ABC}{(1+\varrho)}-ABC\right]=s\times\frac{1-\varrho ABC}{(1+\varrho)}$: and though this case is confined to three lives, yet it is easy to see that the method of solution is general, and will apply to any other number of lives; whence the following rule.

§ 180. *Multiply the value of an annuity on the given lives by the rate of interest, and subtract the product from unity; divide the remainder by the amount of £1 in one year, and the quotient multiplied by the given sum will be the value required.*

For examples of the use and application of this problem, see Question XXVII in Chapter XII.

COROLLARY I.

§ 181. In this problem I have considered the present value of the reversionary sum as depending on the extinction of the given lives in *whatever* year that may happen during the probability of their joint continuance: but, if this contingency is *Deferred* for any number of years ($=n$); that is, if we wish to ascertain the present value of a given sum payable on the failure of such lives, provided that shall happen *after* the given period, the formula will be materially altered. For, by pursuing the same steps as in the problem, it will be found that the expectation of receiving the given sum, at the end of the $(n+1)^{\text{st}}$, $(n+2)^{\text{nd}}$, $(n+3)^{\text{rd}}$, &c, years, to the utmost extent of human life, will be denoted by the series $s \times \left[\frac{\alpha\beta\gamma - \overset{\prime}{\alpha}\overset{\prime}{\beta}\overset{\prime}{\gamma}}{abc(1+\varrho)^{n+1}} + \frac{\overset{\prime}{\alpha}\overset{\prime}{\beta}\overset{\prime}{\gamma} - \overset{\prime\prime}{\alpha}\overset{\prime\prime}{\beta}\overset{\prime\prime}{\gamma}}{abc(1+\varrho)^{n+2}} + \frac{\overset{\prime\prime}{\alpha}\overset{\prime\prime}{\beta}\overset{\prime\prime}{\gamma} - \overset{\prime\prime\prime}{\alpha}\overset{\prime\prime\prime}{\beta}\overset{\prime\prime\prime}{\gamma}}{abc(1+\varrho)^{n+3}} + \&c\right]$: which may be more conveniently divided into the two following series,

$$\frac{s}{abc(1+\varrho)^{n}} \times \left[\frac{\alpha\beta\gamma}{(1+\varrho)} + \frac{\overset{\prime}{\alpha}\overset{\prime}{\beta}\overset{\prime}{\gamma}}{(1+\varrho)^{2}} + \frac{\overset{\prime\prime}{\alpha}\overset{\prime\prime}{\beta}\overset{\prime\prime}{\gamma}}{(1+\varrho)^{3}} + \&c\right] -$$

$$\frac{s}{abc(1+\varrho)^{n}} \times \left[\frac{\overset{\prime}{\alpha}\overset{\prime}{\beta}\overset{\prime}{\gamma}}{(1+\varrho)} + \frac{\overset{\prime\prime}{\alpha}\overset{\prime\prime}{\beta}\overset{\prime\prime}{\gamma}}{(1+\varrho)^{2}} + \frac{\overset{\prime\prime\prime}{\alpha}\overset{\prime\prime\prime}{\beta}\overset{\prime\prime\prime}{\gamma}}{(1+\varrho)^{3}} + \&c\right].$$

The latter of these is, by Prob. I, cor. 3, equal to $-s(ABC)^{\mathrm{d}}$; and the former is equal to $s \times \frac{\frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n} + (ABC)^{\mathrm{d}}}{(1+\varrho)}$: consequently the required present value of the deferred assurance will be equal to $s \times \frac{\frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n} - \varrho(ABC)^{\mathrm{d}}}{(1+\varrho)}$; and whence the following rule.*

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* Since $(ABC)^{\mathrm{d}}$ is, by Prob. I, cor. 3, equal to $A^{\circ}B^{\circ}C^{\circ} \times \frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n}$; it is obvious that the present value, in the case of *single* or *joint* lives, might be more conveniently expressed by $s \times \frac{1-\varrho A^{\circ}B^{\circ}C^{\circ}}{(1+\varrho)} \times \frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n}$: and it is from this formula that I have deduced the rule in Question XXVIII, Chapter XII. But that rule will not extend to *all* cases.

§ 182. *Multiply the value of a Deferred annuity on the given lives, by the rate of interest; subtract the product from the expectation that the given lives shall receive £1 at the end of the given term; and divide the difference by the amount of £1 in a year: the quotient thence arising, being multiplied by the given sum, will produce the value required.*

For examples of the use and application of this corollary, see Question XXVIII in Chapter XII.

COROLLARY II.

§ 183. But, if the contingency is *Temporary*; that is, if we wish to ascertain the value of a given sum payable on the failure of such lives, provided that shall happen *within a given term* $(=n)$, it will be found by taking the first n terms of the series given in the problem. Whence, such present value will in this case be denoted by the series $s \times \left[\frac{abc - a'b'c'}{abc(1+\varrho)} + \frac{a'b'c' - a''b''c''}{abc(1+\varrho)^2} + \frac{a''b''c'' - a'''b'''c'''}{abc(1+\varrho)^3} + \ldots\ldots \frac{\underset{\prime}{\alpha}\underset{\prime}{\beta}\underset{\prime}{\gamma} - \alpha\beta\gamma}{abc(1+\varrho)^n}\right]$: which may be more conveniently divided into the two following ones; $\frac{s}{abc(1+\varrho)}\left[abc + \frac{a'b'c'}{(1+\varrho)} + \frac{a''b''c''}{(1+\varrho)^2} + \frac{a'''b'''c'''}{(1+\varrho)^3} + \ldots\ldots\ldots \frac{\underset{\prime}{\alpha}\underset{\prime}{\beta}\underset{\prime}{\gamma}}{(1+\varrho)^{n-1}}\right] - \frac{s}{abc}\left[\frac{a'b'c'}{(1+\varrho)} + \frac{a''b''c''}{(1+\varrho)^2} + \frac{a'''b'''c'''}{(1+\varrho)^3} + \ldots\ldots\ldots \frac{\alpha\beta\gamma}{(1+\varrho)^n}\right]$. But, the

first of these is equal to $s \times \frac{1 + (ABC)^{\mathrm{t}-1}}{(1+\varrho)}$;[*] and the second, which is to be subtracted, is evidently equal to $s(ABC)^{\mathrm{t}}$: consequently the required value of the assurance *for the given term* might be denoted by the formula $s\left[\frac{1+(ABC)^{\mathrm{t}-1}}{(1+\varrho)} - (ABC)^{\mathrm{t}}\right]$.

But, since $(ABC)^{\mathrm{t}-1} + \frac{\alpha\beta\gamma}{abc(1+\varrho)^n}$ is equal to $(ABC)^{\mathrm{t}}$,[†] whereby $(ABC)^{\mathrm{t}-1}$ becomes equal to $(ABC)^{\mathrm{t}} - \frac{\alpha\beta\gamma}{abc} \times (1+\varrho)^{-n}$; and since, by Prob. I, cor. 4, $(ABC)^{\mathrm{t}}$ is equal to $ABC - (ABC)^{\mathrm{d}}$; we may render the above formula more convenient for general use by substituting such value therein instead of $(ABC)^{\mathrm{t}}$: whereby it will become equal to $s \times \frac{1 - \varrho ABC - \left[\frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n} - \varrho(ABC)^{\mathrm{d}}\right]}{(1+\varrho)}$: and whence the following rule.

§ 184. *From the present value of the assurance of the given sum payable on the extinction of the*

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[*] The new character $(ABC)^{\mathrm{t}-1}$ denotes the value of a temporary annuity on the joint lives ABC for $n-1$ years only, and is introduced here merely to show the steps of the process; and because the rules, given by Mr. Morgan, for the solution of this and other similar questions, are founded on one of these incomplete steps. See Price's *Obs. on Rev. Pay.* vol. i. p. 53, note *x*; and p. 61, note *e*.

[†] As is evident by continuing the series which this quantity expresses.

given lives, subtract the present value of the same assurance Deferred for the given term: the difference will be the value required.[*]

For examples of the use and application of this corollary, see Question XXIX in Chapter XII.

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[*] We may oftentimes obtain a *near* value of an assurance for a give *term* (particularly when only one life, A, is concerned) by the following simple method. The series expressing such value is, by the corollary, equal to $\frac{s}{a} \times \left[\frac{a - \overset{\prime}{a}}{(1+\varrho)} + \frac{\overset{\prime}{a} - \overset{\prime\prime}{a}}{(1+\varrho)^2} + \frac{\overset{\prime\prime}{a} - \overset{\prime\prime\prime}{a}}{(1+\varrho)^3} + \ldots\ldots\ldots\ldots \frac{\alpha_{\prime} - \alpha}{(1+\varrho)^{n}} \right]$: now, if we suppose the decrements of life to be equal and uniform throughout the given term (which is usually the case in the middle ages of life, particularly if the Northampton tables are made use of), and if such decrements be denoted by δ; then will this series become $\frac{s}{a}\left[\frac{\delta}{(1+\varrho)} + \frac{\delta}{(1+\varrho)^2} + \frac{\delta}{(1+\varrho)^3} + \ldots\ldots \ldots\ldots \frac{\delta}{(1+\varrho)^{n}} \right] = \frac{s \,.\, \delta}{a} \times \frac{1-(1+\varrho)^{-n}}{\varrho}$. But, since δ should in all those cases where the decrements are not *exactly* uniform be taken equal to $\frac{a-\alpha}{n}$, this formula may be more conveniently expressed by $s \times \frac{a-\alpha}{an} \times \frac{1-(1+\varrho)^{-n}}{\varrho}$: whence the following rule.

Divide the probability that the life shall fail in the given term by the number of years; multiply the quotient by the present value of an annuity for the given term, and also by the given sum: the product will be the value required.

It will readily appear that, if the given term is wholly within the limits of equal decrements, the above formula will give the *exact* value of the assurance for such term; and that if the decre-

COROLLARY III.*

§ 185. Though in this problem the investigation is confined to the case of three joint lives, yet it is easy to apply the same method of reasoning to the case of any single life, or to any number of joint lives, or to the longest of any number of lives, or to any number of lives out of any other number of lives, &c. And it will be found that by substituting the value of an annuity on such single life, on such joint lives, on such longest lives, &c, &c, instead of the quantity denoted by ABC in the given formula in § 179, we shall obtain the true values in such cases accordingly.

§ 186. The same remark will also apply to the two preceding corollaries, if (in addition to the substitution above alluded to) we also substitute the value of similar annuities Deferred for the given term, instead of the quantity $(ABC)^{\mathrm{d}}$ in either formula in

ments are not exactly uniform, but nearly so, it will not materially differ from the true value of such assurance. The formula, however, must always be used with proper caution, and with reference to the table of observations employed; as it is not suited to every case.

The same observations will apply to the case of *joint* lives, if δ be taken equal to $\frac{abc-\alpha\beta\gamma}{n}$: but we are not so likely, in practice, to meet with cases of assurances on joint lives for terms, where such terms fall within the limits of equal decrements.

* Simpson, Prob. 22, and *Sup.* Prob. 27. Morgan, Prob. 8, cor. and Prob. 9, cor.

§ 181 or 183. But here it should be particularly observed that, when a Deferred or Temporary assurance depends on the *longest* of any number of lives, the quantity $\frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n}$ in the given formula will denote the expectation that the *longest* of such lives will receive £1 at the end of the given term: and that, when such assurance depends on the extinction of any *two out of three* lives, the same quantity will denote the expectation that any *two out of three* lives will receive £1 at the end of the given term: &c. &c.

For examples of the use and application of this corollary, see Questions XXVII, XXVIII and XXIX in Chapter XII.

PROBLEM XXIII.*

§ 187. To find the value of an *annuity* to commence at the end of the year† in which any number of lives become extinct.

SOLUTION.

Let us, as in the preceding problem, confine the case to three joint lives ABC, whose probabilities of continuing 1, 2, 3, &c, years are as denoted in § 24: and let the annuity be a perpetuity, or estate in fee. Now, the chance which the heir of this estate has of receiving the rent at the end of any one year will de-

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[*] De Moivre, Prob. 6 and 29. Dodson, vol. ii. Ques. 88.

[†] That is, the *first* payment of the annuity is to be then made.

pend altogether on the joint lives becoming extinct before the end of that year. The probability of this event happening before the end of the first year is $\left(1-\frac{a'b'c'}{abc}\right)$, which being multiplied by $(1+\varrho)^{-1}$ will give the value of his expectation of receiving the first year's rent. In like manner, the probability of the joint lives failing before the end of the second year is $\left(1-\frac{a''b''c''}{abc}\right)$, which being multiplied by $(1+\varrho)^{-2}$ will give the value of his expectation of receiving the second year's rent. By a similar method of reasoning it will be found that $\left(1-\frac{a'''b'''c'''}{abc}\right)$ multiplied by $(1+\varrho)^{-3}$ will give the value of his expectation of receiving the third year's rent. And so on, *ad infinitum:* for, since the estate must eventually revert to the heir, his expectation will never cease. Consequently the sum of all those values continued to infinity will be the total present value of the estate to be enjoyed after the extinction of the joint lives ABC: or, in other words, it will be that sum which a person ought to give to have it assured to him after the extinction of those lives.

§ 188. But the sum of the above terms is equal to the two following series; $\frac{1}{(1+\varrho)}+\frac{1}{(1+\varrho)^2}+\frac{1}{(1+\varrho)^3}+$ .... *ad infinitum* $-\frac{1}{abc}\times\left[\frac{a'b'c'}{(1+\varrho)}+\frac{a''b''c''}{(1+\varrho)^2}+\frac{a'''b'''c'''}{(1+\varrho)^3}+\&c\right]$. The former of these is equal to $\frac{1}{\varrho}$, or the present value of the perpetuity of an annuity of

£1 per annum: but the latter, which is to be subtracted, will cease on the extinction of the oldest life, and is therefore equal to the value of an annuity on the joint lives ABC. Consequently the total value of the reversionary annuity becomes equal to $\frac{1}{\varrho} - ABC$: and though this case is confined to that of three joint lives, yet it is easy to see that the method of solution will equally apply to that of any other number of lives; whence the following rule.

§ 189. *Subtract the value of an annuity on the given lives from the present value of the estate or given annuity; the difference will be the value of the reversionary estate or annuity required.*

For examples of the use and application of this problem, see Question XXVI in Chapter XII.

COROLLARY I.

§ 190. If the annuity, instead of being a perpetuity, is only for a number of years ($=n$) *greater* however than that to which it is probable the given lives may extend,* we must substitute the value of such terminable annuity instead of the value of the perpetuity above mentioned: whereby the formula becomes $\frac{1-(1+\varrho)^{-n}}{\varrho} - ABC$; and consequently the rule above given is also applicable to the present case.

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* The term to which it is probable that any given life or lives may extend is—for a *single* life, equal to the difference between the age of such life and the age of the oldest life in the table of observations:—for *joint* lives, equal to the difference between the *oldest* of such lives and the age of the oldest life in the table:—for the *longest* of any lives, equal to the difference between the age of the *youngest* of such lives and the age of the oldest life in the table.

For examples of the use and application of this corollary, see Question XXVI in Chapter XII.

COROLLARY II.*

§ 191. But, if the annuity (instead of being a perpetuity, or for any long period) is for a term of years $(=n)$ which is *less* than that to which it is probable the given lives may extend; the series, expressing the required value in the problem, will terminate at the end of that period; and will consequently become equal to $\frac{1}{(1+\varrho)} + \frac{1}{(1+\varrho)^2} + \frac{1}{(1+\varrho)^3} + \ldots\ldots \frac{1}{(1+\varrho)^n} - \frac{1}{abc}\left[\frac{a'b'c'}{(1+\varrho)} + \frac{a''b''c''}{(1+\varrho)^2} + \frac{a'''b'''c'''}{(1+\varrho)^3} + \ldots\ldots \frac{\alpha\beta\gamma}{(1+\varrho)^n}\right]$. But, the first part is equal to $\frac{1-(1+\varrho)^{-n}}{\varrho}$; and the latter part is, by Prob. I, cor. 4, equal to $-(ABC)^{\text{t}}$: whence the following rule.

§ 192. *From the value of an annuity certain for the given term, subtract the value of a Temporary annuity on the given lives for the given term: the difference will be the value required.*

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* Simpson's *Sup.* Prob. 14. De Moivre, Prob. 28. Morgan, Prob. 4, cor. 2. Dodson, vol. ii. Ques. 92.

COROLLARY III.

§ 193. If the contingency (on which the payment of the annuity mentioned in the problem depends) is *Deferred* for a given number of years ($=n$); the series expressing the general value of the reversion of the perpetuity must commence at the end of that period, and be continued on to infinity: that is, the series $\frac{1-\frac{\overset{\prime}{\alpha}\overset{\prime}{\beta}\overset{\prime}{\gamma}}{\alpha\beta\gamma}}{(1+\varrho)^{n+1}}+\frac{1-\frac{\overset{\prime\prime}{\alpha}\overset{\prime\prime}{\beta}\overset{\prime\prime}{\gamma}}{\alpha\beta\gamma}}{(1+\varrho)^{n+2}}+\frac{1-\frac{\overset{\prime\prime\prime}{\alpha}\overset{\prime\prime\prime}{\beta}\overset{\prime\prime\prime}{\gamma}}{\alpha\beta\gamma}}{(1+\varrho)^{n+3}}+\ldots$ *ad infinitum,* will denote the value of the required reversionary annuity, to be entered on provided either of the given lives become extinct after the n^{th} year. Now this series may be resolved into two others, namely, $(1+\varrho)^{-n}\times\left[\frac{1}{(1+\varrho)^1}+\frac{1}{(1+\varrho)^2}+\frac{1}{(1+\varrho)^3}+\ldots\right.$ ..... *ad infinitum*$\left.\right]-\frac{1}{\alpha\beta\gamma(1+\varrho)^n}\left[\frac{\overset{\prime}{\alpha}\overset{\prime}{\beta}\overset{\prime}{\gamma}}{(1+\varrho)}+\frac{\overset{\prime\prime}{\alpha}\overset{\prime\prime}{\beta}\overset{\prime\prime}{\gamma}}{(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{\alpha}\overset{\prime\prime\prime}{\beta}\overset{\prime\prime\prime}{\gamma}}{(1+\varrho)^3}+\&c\right]$: the first of which is equal to $(1+\varrho)^{-n}\times\frac{1}{\varrho}$; and the latter is, by Prob. I, cor. 3, equal to $-(1+\varrho)^{-n}\times A^{\circ}B^{\circ}C^{\circ}$. Consequently, the value of the reversionary annuity would be equal to $(1+\varrho)^{-n}\times\left[\frac{1}{\varrho}-A^{\circ}B^{\circ}C^{\circ}\right]$ were it certain that the given lives would continue in being to the end of n years: but, as the probability of this event is $\frac{\alpha\beta\gamma}{abc}$, we must multiply the above expression by this quantity, in order to obtain the total present value of the same. Whence the true value will be equal to $\frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n}\times\frac{1}{\varrho}-A^{\circ}B^{\circ}C^{\circ}\times\frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n}$. But $A^{\circ}B^{\circ}C^{\circ}\times\frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n}$ is, by Prob. I, cor. 3, equal to $(ABC)^{\mathrm{d}}$; therefore the above value becomes $\frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n}\times\frac{1}{\varrho}-(ABC)^{\mathrm{d}}$: and whence the following rule.*

§ 194. *Multiply the expectation of the given lives receiving £1 at the end of the given term by the perpetuity; from the product subtract the value of an annuity on the given lives Deferred for the given term: the difference will be the value required.*

COROLLARY IV.†

§ 195. Or, if the contingency is *Temporary*; that is, if it be required to ascertain the value of such annuities depending on the extinction of any given lives, provided that shall happen *within a given period* $(=n)$; the value of the assurance will be expressed by the series $\frac{1-\frac{\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}}{(1-\varrho)}+\frac{1-\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}}{(1+\varrho)^2}+\frac{1-\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}}{(1+\varrho)^3}+\ldots\ldots\frac{1-\frac{\alpha\beta\gamma}{abc}}{(1+\varrho)^n}+\frac{1-\frac{\alpha\beta\gamma}{abc}}{(1+\varrho)^{n+1}}+\frac{1-\frac{\alpha\beta\gamma}{abc}}{(1+\varrho)^{n+2}}+$ *ad infinitum*. For, the n^{th} year's rent *and all the subsequent ones* being dependent on the given lives becoming extinct in n years, it is obvious that all such subsequent ones must be multiplied by the common factor $\left(1-\frac{\alpha\beta\gamma}{abc}\right)$. But, the first n terms of this series are equal to $\frac{1-(1+\varrho)^{-n}}{\varrho}$ $-(ABC)^{\text{t}}$; as found by the second corollary: and the remaining terms are equal to $\left(1-\frac{\alpha\beta\gamma}{abc}\right)\times(1+\varrho)^{-n}\times\left[(1+\varrho)^{-1}+(1+\varrho)^{-2}+(1+\varrho)^{-3}+\ldots\ldots\ \textit{ad infinitum}\right]=\left(1-\frac{\alpha\beta\gamma}{abc}\right)\times(1+\varrho)^{-n}$ $\times\frac{1}{\varrho}$. Consequently the total value of the series becomes $\frac{1-(1+\varrho)^{-n}}{\varrho}-(ABC)^{\text{t}}+\frac{(1+\varrho)^{-n}}{\varrho}-\frac{\alpha\beta\gamma}{abc}$ $(1+\varrho)^{-n}\times\frac{1}{\varrho}$: which, since $(ABC)^{\text{t}}=ABC-(ABC)^{\text{d}}$, may be further reduced to $\frac{1}{\varrho}-ABC-\left[\frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n}\times\frac{1}{\varrho}-(ABC)^{\text{d}}\right]$: and whence the following rule.*

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* See the note in page 147.

† Simpson, Prob. 23; and *Sup.* Prob. 31. Price, Note (G). Dodson, vol. ii. Ques. 93.

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* If the annuity is to be entered upon *at the end of the given time* (in case of the failure of the given lives), and not *at the end of the year in which such failure may happen,* its present value will be equal to $\left(1-\frac{\alpha\beta\gamma}{abc}\right)\times(1+\varrho)^{-n}\times\left(1+\frac{1}{\varrho}\right)=\left(1-\frac{\alpha\beta\gamma}{abc}\right)\frac{(1+\varrho)^{-(n-1)}}{\varrho}$: for, such reversionary perpetuity is evidently to be entered on and enjoyed at the end of the n^{th} year, provided the given lives fail prior thereto. And care should be taken not to confound this case with the one mentioned in the text.

§ 196. *From the whole value of the reversion of the perpetuity after the given lives, subtract the value of the same Deferred for the given term: the difference will be the value required.*

COROLLARY V.

§ 197. It will be obvious, from what has been said in the first corollary, that if the annuity, instead of being a perpetuity, is for a given term only, the rules given in the two preceding corollaries will still be correct; provided we substitute, for the *perpetuity*, the value of an annuity for the *given term.* Such terminable annuity however must always be for a term of years *greater* than that, for which the assurance is made.

COROLLARY VI.

§ 198. Though the above problem and its corollaries are confined, in the investigation, to the case of three joint lives, yet it is easy to apply the same method of reasoning to the case of any single life, or to any number of joint lives, or to the longest of any number of lives,* or to any number of lives out of any other number of lives, &c, &c, either for the *whole* lives or for a given *term.*† And it will be found that by substituting the value of an annuity on such single life, on such joint lives, on the longest of any number of lives, &c, (either for the whole life or Deferred for the given term) instead of the value ABC or $(ABC)^{\mathrm{d}}$ in the given formulæ, we shall obtain the true values in such cases accordingly.

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* Dodson, vol. ii. Ques. 90, 91.

† Simpson, Prob. 24.

§ 199. It should be particularly observed however that, when the *longest* of any number of lives are concerned, the quantity $\frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n}$ will represent the expectation that the *longest* of such lives will receive £1 at the end of the given term. And a similar remark will apply when the assurance depends on the extinction of any *two out of three* lives: &c. &c. See Problem XXII, cor. 3.

SCHOLIUM.

§ 200. From these two problems and their corollaries may be determined all questions relative to the value of assurances of any given sum, or of any given annuity, depending on the contingencies therein stated: and as any given sum may be converted into a corresponding perpetuity (or into a perpetual annuity of a corresponding value) by multiplying it by the interest of £1 for a year, it would seem at first to be the same thing to determine the present value of a given *sum* depending on the extinction of any lives, as to determine the value of a corresponding *annuity* for ever, depending on the same lives: that is, it would appear to be the same thing to determine the present value of £100 payable at the end of the year in which any given lives became extinct, as to determine the value of an estate yielding £5 per annum, and to be entered upon at the same period; interest being reckoned at 5 per cent. But, it should be observed that the first yearly payment of a reversionary *annuity* becomes due and is payable at the end of the year in which the lives fail; however much or little of that year may then happen to be unexpired: and this likewise is the time when a reversionary *sum* becomes due. The expectant therefore in the former case will have entered on his annuity, or received the first year's rent of it, at the very time that the expectant of the sum is supposed to have laid out such sum in the purchase of a perpetuity of a corresponding value; the first year's rent of which he will not receive till the *end of the following year.* Consequently, a reversionary estate is worth one year's purchase more than a corresponding reversionary sum: whence the former is to the latter in the ratio of £1, increased by its interest for one year, to £1; that is, as $(1+\varrho)$ to 1.

§ 201. Therefore, if the present value of a reversionary sum be multiplied by $(1+\varrho)$ it will give the value of a corresponding reversionary estate or annuity. Thus, the present value of a reversionary annuity of £1 per annum for ever after the extinction of the joint lives ABC is, by Prob. XXIII, equal to $\frac{1-\varrho ABC}{\varrho}$; and the present value of a corresponding reversionary sum $\left(=\frac{1}{\varrho}\right)$ after the same lives, is, by Prob. XXIII, equal to $\frac{1}{\varrho}\times\frac{1-\varrho ABC}{(1+\varrho)}=\frac{1}{(1+\varrho)}\times\frac{1-\varrho ABC}{\varrho}$: but, it is evident on inspection that the former value is to the latter in the ratio of $(1+\varrho)$ to 1.

§ 202. Hence it is evident also that the value of any reversionary annuity after any given lives, being divided by the amount of £1 in a year, will give the present value of a corresponding reversionary sum after the same lives.*

These remarks may be more fully confirmed by any of the similar examples in the two problems, or their corollaries: as may be seen in the Scholium to Question XXVII in Chapter XII.

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* Mr. Simpson not having attended to this circumstance, it becomes necessary to correct the rules given by him for the solution of Prob. 21 and 22 in his *Doctrine of Annuities, &c;* and of Prob. 26, 27, 32, 33, and others of a similar kind, in his *Supplement.* The same observation will apply to the problems of M. De Moivre and Mr. Dodson, alluded to in the note in page 144.

CHAPTER VII.

On Successive Life Annuities and Copyhold Estates.

§ 203. In all the preceding problems, involving the question of reversionary life annuities, the lives of the expectants have been supposed to be such as are *now* fixed on and determined; and the value of an annuity on their lives consequently becomes less and less, according as their period of coming into possession might be prolonged. In such questions, however, as relate to the present division of the subject, the life which is to succeed to the annuity, after the extinction of the life in possession, is supposed to be one which is *then* to be fixed on at pleasure; and which will probably be one of the best lives that can then be found. This life, therefore, may be considered as having a fixed and determinate value, since such a life may generally be chosen as will best answer the views of persons concerned in questions of this kind; and it is usual to conceive a *mean* age at which they are all admitted. But, the nature of the cases, which involve the consideration of this subject, will best appear from the following problems.

PROBLEM XXIV.*

§ 204. Supposing A to enjoy an annuity for his life; and, at his decease, to have the nomination of a successor, B, who is also to enjoy the annuity for his life: To find the present value of the annuity on the succeeding life; and also the value of the two successive lives.

SOLUTION.

Let the succeeding life B, to be put in nomination at the decease of A, be such that an annuity on his life *at that time* may be equal to B. Now, since the probability that the first life fails, or that the second comes into possession, in the first year is $\frac{a-\overset{\prime}{a}}{a}$; and as the total value of what the second life will be entitled to, on the happening of this event, is B, it follows that his expectation of coming into possession the first year (or of receiving an equivalent sum, equal to B) will, as in Prob. XXII, be denoted by $B \times \frac{a-\overset{\prime}{a}}{a(1+\varrho)}$. In like manner, it will be found that his expectation of coming into possession in the second year (or of receiving an equivalent sum B) will be denoted by $B \times \frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{a(1+\varrho)^2}$: and that his expectation of coming into possession in the third year (or of receiving the equivalent sum B) will be denoted by $B \times \frac{\overset{\prime\prime}{a} - \overset{\prime\prime\prime}{a}}{a(1+\varrho)^3}$: and so on for every succeeding year to the utmost extent of A's life.

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* Simpson, Prob. 25, and *Sup*. Prob. 24. De Moivre, Prob. 13. Morgan, Prob. 14.

But the sum of all these values is equal to $\frac{B}{a(1+\varrho)} \times \left[a + \frac{\overset{\prime}{a}}{(1+\varrho)} + \frac{\overset{\prime\prime}{a}}{(1+\varrho)^2} + \frac{\overset{\prime\prime}{a}}{(1+\varrho)^3} + \&c\right] - \frac{B}{a} \times \left[\frac{\overset{\prime}{a}}{(1+\varrho)} + \frac{\overset{\prime\prime}{a}}{(1+\varrho)^2} + \frac{\overset{\prime\prime\prime}{a}}{(1+\varrho)^3} + \&c\right]$; which, by Prob. XXII, is equal to $B \times \frac{1-\varrho A}{(1+\varrho)}$; and which would be the value required, were B in reality the value of a reversionary *sum* to be received on the decease of A: but since it denotes the value of an *annuity*, the first payment of which commences at the end of the year in which the life A fails, we must multiply the above expression by $(1+\varrho)$, agreeably to what has been said in the scholium in page 160; whence, the true present value of the successive life will be $B(1-\varrho A)$; and whence the following rule.

§ 205. *Multiply the value of an annuity on the life in possession by the rate of interest, and subtract the product from unity; multiply the remainder by the assumed value of an annuity on the succeeding life: the product will be the present value of an annuity on such succeeding life.*

§ 206. If this present value be added to the value of an annuity on the life in possession, it will give

$A + B(1 - \varrho A)$ for the value of the *two* successive lives.

For examples of the use and application of this problem see Question XXIII in Chapter XII.

COROLLARY I.

§ 207. Hence may be determined the present value of an annuity on any number of *joint* lives, or on the *longest* of any number of lives, &c, nominated to succeed after any other number of joint lives, or after the longest of any other number of lives, &c: for, by making A and B, respectively, equal to the value of an annuity on such joint lives, or the longest of such lives, &c, the above formula will express the true present values of such successive lives.

COROLLARY II.*

§ 208. If the succeeding life, instead of receiving an annuity *during his life*, were to receive an annuity *certain for a given term of years* after the failure of the life in possession; then, by making B equal to the value of such annuity certain for the given term, the above formula would truly express the present value of such annuity to be entered on at the failure of the life in possession.

And if this annuity were a perpetuity (that is, if the succeeding life and his heirs were to receive an annuity for ever after the failure of the life in possession) B would become equal to $\frac{1}{\rho}$, and the formula would in this case become equal to $\frac{1}{\rho} - A$; which is the very same as that deduced from Prob. XXIII.

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* Morgan, Prob. 13.

For examples of the use and application of this corollary see Question XXIV in Chapter XII.

PROBLEM XXV.*

§ 209. Three lives A,B,C, being given in succession: To find the present value of an annuity on the third succeeding life: and also the value of the three successive lives.

SOLUTION.

Let the values of an annuity on each of the three lives, at the time that they severally come into possession, be respectively denoted by A,B,C. Therefore, since the value of an annuity on the second life in succession (to commence at the decease of A) is to the value of a perpetuity (to commence at the same time) in the ratio of B to $\frac{1}{\rho}$; it follows that the present value of the former will be to the present value of the latter in the same ratio. But, the present value of the latter is, by Prob. XXIII, equal to $\frac{1}{\varrho} - A = \frac{1-\varrho A}{\varrho}$: therefore $\frac{1}{\varrho} : B :: \frac{1-\varrho A}{\varrho} : B(1-\varrho A)$ = the present value of the former; and which is the same value as that found by the last problem: whence the two methods of solution confirm the truth of each other.

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* Simpson, Prob. 26, and *Sup*. Prob. 25.

The present value of the first two successive lives being thus found equal to $A + B(1-\varrho A)$; it follows that the present value of the reversion of a perpetuity after these lives will (by a deduction from Prob. XXIII) be equal to $\frac{1}{\varrho} - \left[(A + B(1-\varrho A)\right] = \frac{1}{\varrho}(1-\varrho A) \times (1-\varrho B)$. Consequently, since the value of an annuity on the third life in succession (to commence at the decease of B) is to the value of a perpetuity (to commence at the same period) in the ratio of C to $\frac{1}{\varrho}$; it follows that the present value of the former will be to the present value of the latter, in the same ratio: that is, $\frac{1}{\varrho} : C :: \frac{1}{\varrho}(1-\varrho A) \times (1-\varrho B) : C \times (1-\varrho A) \times (1-\varrho B)$ = the present value of an annuity on the third successive life: whence the following rule.

§ 210. *Multiply the value of an annuity on the life in possession by the rate of interest, and subtract the product from unity; multiply also the assumed value of an annuity on the second life in succession by the rate of interest, and subtract this product likewise from unity: multiply together these two remainders, and their product again by the assumed value* *of an annuity on the third life in succession: this last product will be the value of the third successive life.*

§ 211. If the present value of each successive life, as above found, be added together, their sum, or $A + B(1-\rho A) + C(1-\rho A) \times (1-\rho B)$, will be the present value of the three successive lives: but this expression will be found equal to $\frac{1}{\rho}\left[1-(1-\rho A) \times (1-\rho B) \times (1-\rho C)\right]$: whence the present value of any number of lives in succession may be discovered on inspection; and thence, the following rule.

§ 212. *Multiply the assumed value of an annuity on each of the proposed lives, by the rate of interest; take the several products from unity, and multiply together all the remainders; let the product thus arising be also subtracted from unity, and the remainder divided by the rate of interest: the quotient will be the present value of all the successive lives, including the life in possession.*

PROBLEM XXVI.*

§ 213. Suppose a person to purchase a copyhold estate, on any number of lives A, B, C, &c, for the sum s, on condition that he and his successors may renew it continually by paying the fine f whenever any one of such lives becomes extinct; To find the present value of the whole purchase of such estate.

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* Simpson, Prob. 27; and *Sup.* Prob. 29. Dodson, vol. iii. Ques. 82 to 95. De Moivre, Prob. 10.

SOLUTION.

Let the value of an annuity on each of the lives A,B,C, &c, in possession, be respectively denoted by A, B, C, *&c:* and let the value of an annuity on each of the lives $\overset{1}{\mathrm{A}}, \overset{2}{\mathrm{A}}, \overset{3}{\mathrm{A}}$, &c, (which are supposed to follow in direct succession from A) be respectively denoted by $\overset{1}{A}, \overset{2}{A}, \overset{3}{A}$, *&c.** In like manner, let the value of an annuity on each of the lives, $\overset{1}{\mathrm{B}}, \overset{2}{\mathrm{B}}, \overset{3}{\mathrm{B}}$, &c, (which are supposed to follow in direct succession from B) be respectively denoted by $\overset{1}{B}, \overset{2}{B}, \overset{3}{B}$, *&c:* and so on with respect to the lives immediately succeeding C, D, &c. And let us first determine the present value of all the fines payable on the extinction of the life A and his immediate successors.

Now, the present value of the fine f payable on the decease of A, in whatever part of the year that may happen, may in the present case be considered equal to the present value of an estate, yielding $f\varrho$ per annum, to be entered upon at the decease of A; which, by Prob. XXIII, is found to be $f\varrho\left(\frac{1}{\varrho} - A\right) = f(1 - \varrho A)$. If instead of A in this formula we substitute the present value of the two successive lives A, $\overset{1}{\mathrm{A}}$, which, by Prob. XXIV, is found equal to $A + \overset{1}{A}(1-\varrho A)$, we shall have $f(1-\varrho A)\times(1-\varrho\overset{1}{A})$ for the present value of the fine to be paid on the decease of $\overset{1}{\mathrm{A}}$. And if instead of A in that same formula we substitute the present value of the three successive lives A, $\overset{1}{\mathrm{A}}$, $\overset{2}{\mathrm{A}}$, which, by the last problem, is found equal to $A+\overset{1}{A}(1-\varrho A)+\overset{2}{A}(1-\varrho A)\times(1-\varrho\overset{1}{A})$, we shall have $f(1-\varrho A)\times(1-\varrho\overset{1}{A})\times(1-\varrho\overset{2}{A})$ for the present value of the fine payable on the decease of $\overset{2}{\mathrm{A}}$: and so on with respect to all the subsequent fines payable on the extinction of each life in direct succession from A. But, the sum of all these quantities, or the series $f\Big[(1-\varrho A)+(1-\varrho A)\times(1-\varrho\overset{1}{A})+(1-\varrho A)\times(1-\varrho\overset{1}{A})\times(1-\varrho\overset{2}{A})+\ldots\ldots\ \textit{ad infinitum}\Big]$ is the present value of all the sums that may be paid from time to time for the renewals of the several lives in direct succession from A.

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* Care must be taken not to mistake these numeral quantities for the *index* of the quantity to which they are annexed: as, they merely denote the order of succession among the given lives.

By a similar method of proceeding it will be found that the series $f\Big[(1-\varrho B)+(1-\varrho B)\times(1-\varrho\overset{1}{B})+(1-\varrho B)\times(1-\varrho\overset{1}{B})\times(1-\varrho\overset{2}{B})+\ldots\ldots\ \textit{ad infinitum}\Big]$ will denote the present value of all the sums that may be paid from time to time for the renewals of the several lives in direct succession from B. And so on with respect to the other lives C,D, &c, and their successors for ever: and the sum of all these different series, or

$$\begin{array}{l}
f\left[(1-\varrho A)+(1-\varrho A)\times(1-\varrho\overset{1}{A})+\dots\ ad\ inf.\right]+\\
f\left[(1-\varrho B)+(1-\varrho B)\times(1-\varrho\overset{1}{B})+\dots\ ad\ inf.\right]+\\
f\left[(1-\varrho C)+(1-\varrho C)\times(1-\varrho\overset{1}{C})+\dots\ ad\ inf.\right]+\\
\qquad\quad \&c \qquad\qquad\qquad \&c \qquad\qquad\qquad \&c,
\end{array}$$

will be the total present value of all the fines that the tenant can ever pay; and which being added to s will give the whole value paid for the purchase.

COROLLARY I.

§ 214. Hence if the lives, with which the lease is from time to time renewed, be supposed equal to one another, or of the same common age $\overset{1}{A}$, the general expression above given will become

$$\begin{array}{l}
f(1-\varrho A)\left[1+(1-\varrho\overset{1}{A})+(1-\varrho\overset{1}{A})^2+(1-\varrho\overset{1}{A})^3+\dots ad\ inf.\right]+\\
f(1-\varrho B)\left[1+(1-\varrho\overset{1}{A})+(1-\varrho\overset{1}{A})^2+(1-\varrho\overset{1}{A})^3+\dots ad\ inf.\right]+\\
f(1-\varrho C)\left[1+(1-\varrho\overset{1}{A})+(1-\varrho\overset{1}{A})^2+(1-\varrho\overset{1}{A})^3+\dots ad\ inf.\right]+\\
\qquad\qquad \&c \qquad\qquad\quad \&c \qquad\qquad\quad \&c \qquad\qquad\quad \&c,
\end{array}$$

the sum of which series, since $\left[1+(1-\varrho\overset{1}{A})+(1-\varrho\overset{1}{A})^2+(1-\varrho\overset{1}{A})^3+\dots\dots ad\ infinitum\right]$ is equal to $\frac{1}{\varrho\overset{1}{A}}$,* becomes $\frac{f}{\overset{1}{A}}\left(\frac{1}{\varrho}-A+\frac{1}{\varrho}-B+\frac{1}{\varrho}-C+\&c\right)$

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* It is well known that $\frac{1}{1-x}=1+x+x^2+x^3+\dots\dots$ *ad infinitum:* therefore, by substituting $(1-\varrho\overset{1}{A})$ for x, we shall have $\frac{1}{\varrho\overset{1}{A}}$ equal to the series given in the text.

$=\frac{f}{\overset{1}{A}}\left(\frac{n}{\varrho}-A-B-C-\&c\right)$; where n denotes the number of lives on which the estate is held: whence the following rule.

§ 215. *Divide the number of lives by the rate of interest, and from the quotient subtract the sum of the values of an annuity on each of the single lives in possession; divide the remainder by the assumed value of an annuity on the common life with which the lease is from time to time to be renewed: the quotient, thence arising, multiplied by the fine to be paid on renewing will be the total present value of all the renewals for ever: and which, being added to the sum given for the estate, will give the whole value of the purchase.*

§ 216. *Example.* Suppose a person to have paid down £1000 for the purchase of a copyhold estate held on three lives, whose ages are 30, 50 and 70; on condition that he may, on the extinction of any life, continually renew with any other life that he thinks proper, on paying a fine of £600: What is the present value of all those fines with which the estate may be continually renewed, reckoning interest at 5 per cent, and the probabilities of living according to the observations of *M. De Parcieux?*

Here we shall have $A=14{\cdot}693$, $B=11{\cdot}363$, $C=6{\cdot}055$, $n=3$, $\varrho={\cdot}05$, $f=600$, and $\overset{1}{A}$ (or the value of an annuity on the best life in the table*) $= 16{\cdot}213$. Consequently, $\frac{600}{16{\cdot}213}(60 - 32{\cdot}111) = 1402{\cdot}171$, or £1402 : 3 : 5 will be the present value of all the fines: and which being added to the £1000 paid upon entering will give the total value of the fee simple of the estate.

COROLLARY II.

§ 217. When all the lives in possession are of the same common age A, the formula in the preceding corollary will become equal to $\frac{fn}{\overset{1}{A}}\left(\frac{1}{\varrho} - A\right)$. But, if all the lives, as well those in possession as those to be put in nomination afterwards, be equal to each other, or of the same common age $\overset{1}{\mathrm{A}}$, the present value of all the renewals for ever will then be equal to $\frac{fn}{\overset{1}{A}}\left(\frac{1}{\varrho} - \overset{1}{A}\right) = fn\left(\frac{1}{\varrho \overset{1}{A}} - 1\right)$.

COROLLARY III.

§ 218. If the present value found by either of the preceding corollaries be multiplied by the rate of interest, it will show how much the rent-roll of the landlord's estate ought in each case to be increased on account of the fines paid at renewing.

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* This is always the assumed value of $\overset{1}{\mathrm{A}}$, agreeably to what has been said on this subject in § 203.

Thus, in the example given in § 216, it will be found that $1402{\cdot}171 \times {\cdot}05 = 70{\cdot}109$, or £70 : 2 : 2 is the sum by which the rent-roll of the lord's estate ought to be increased on account of the fines there mentioned.

COROLLARY IV.

§ 219. Since the purchase money paid for the lease, together with the present value of all the fines to be paid on renewal, is equal to the value of the perpetuity of the rack-rent of the estate $(= p)$; that is, since $s + \frac{f}{\overset{2}{A}}\left(\frac{n}{\varrho} - A - B - C - \&c\right) = p$; it follows that f, or the fine which ought to be paid on renewing, will be equal to $\frac{(p-s)\overset{1}{A}}{\frac{n}{\varrho} - A - B - C - \&c}$: whence the following rule.

§ 220. *Subtract the tenant's interest in the lease (or the purchase money which he has given for the same) from the value of the fee simple of the estate, and multiply the remainder by the assumed value of an annuity on the common life with which the lease is supposed to be constantly renewed; reserving the product:—divide the number of lives on which the lease is now held, by the rate of interest; and from the quotient subtract the sum of the values of an annuity on each of those lives: the reserved product being divided by this remainder, will give the sum which ought in justice to be paid as a fine on each renewal.*

§ 221. *Example.* Suppose a person to have purchased, for £1000, a copyhold estate, the rack-rent of which is estimated at £100 per annum; and that such estate is held on three lives, renewable for ever on the extinction of either of those lives by paying a fine certain: what ought such fine to be fixed at, in order that the purchaser may make 5 per cent interest of his money, supposing the ages of the lives (on which the estate is now held) to be 30, 50 and 70, and that the probabilities of living are according to the observations of *M. De Parcieux?*

Here we shall have p (or the value of the perpetuity of the rack-rent of the estate) $= 2000$, $s = 1000$, and the remaining quantities as in the example in § 216. Consequently the value of the fine ought to be $\frac{1000 \times 16{\cdot}213}{60 - 32{\cdot}111} = 581{\cdot}340$, or £581 : 6 : 10.

COROLLARY V.

§ 222. If the estate is held on one life only, the present value of the landlord's interest therein will be *universally* expressed by $f \times \frac{1 - \varrho A}{\varrho \overset{1}{A}}$. Now, *immediately after* the receipt of a fine, the life in possession is equal to $\overset{1}{A}$; whence the expression in this case becomes $f \times \frac{1 - \varrho \overset{1}{A}}{\varrho \overset{1}{A}}$: and *immediately before* the receipt of a fine, the life in possession having become extinct, the expression in this case becomes $f \times \frac{1}{\varrho \overset{1}{A}}$.*

* These three formulæ will serve to express the value of perpetual *Advowsons* (considered as an object of traffic) under the

But, since the present value of the next fine is universally expressed by $f(1-\varrho A)$, or by the amount of such fine multiplied into the difference between unity and the product of the rate of interest by the value of an annuity on the life in possession, we may readily determine the landlord's interest in the estate, or the value of all the fines to be paid on renewing, by the following rule.

§ 223. *Divide the present value of the next fine by the product of the rate of interest into the value of an annuity on the common life with which the lease is to be continually renewed: the quotient thence arising will be the value required.*

three most usual circumstances: 1° where the living is *possessed by an incumbent* whose life is equal to A; 2° immediately *after presentation*, where the life presented is of the same common age, $\overset{1}{A}$, as that with which the living is supposed to be constantly filled up; 3° with *immediate resignation*.

It must however be here particularly observed that, in *these* cases the value of $\overset{1}{A}$ (or the value of an annuity on the common life with which the living is supposed to be constantly filled up) must never be assumed so great as in those cases mentioned in the text: because the person, who is presented to the living, must always be above 24 years of age: and it seldom happens that he is even so young as this.

It has been ingeniously suggested that the ages of the incumbents, when they are inducted, may be partly fixed from the value of the livings. See De Moivre, Prob. 14 and 15: and Dodson, vol. iii. page 347.

§ 224. *Example.* Suppose that a copyhold tenant pays to the lord of the manor a fine of £100 on his admission, and that every successor does the same; what is the present value of the lord's interest in that copyhold, on the supposition that the tenants admitted thereto are (one with another) 25 years of age at the time of their admission: interest being reckoned at 5 per cent, and the probabilities of living as at *Northampton?*

Here we shall have $\overset{1}{A} = 13{\cdot}567$, $\rho = {\cdot}05$, and $f = 100$; consequently the value of the lord's interest *immediately before* the receipt of a fine will be equal to $100 \times \frac{1}{{\cdot}05 \times 13{\cdot}567} = 147{\cdot}416$, or £147 : 8 : 4: and *immediately after* the receipt of a fine it will be equal to $100 \times \frac{1 - {\cdot}05 \times 13{\cdot}567}{{\cdot}05 \times 13{\cdot}567} = 47{\cdot}416$ or £47 : 8 : 4. But if the life now in possession be 70 years of age, we shall have $A = 6{\cdot}023$; in which case, the lord's interest will be $100 \times \frac{1 - {\cdot}05 \times 6{\cdot}023}{{\cdot}05 \times 13{\cdot}567} = 103{\cdot}022$, or £103 : 0 : 5.

Therefore if the tenant gave £500 for the lease, the whole value of the purchase may in this latter case be estimated at £603; and the corresponding rent at £30 : 3 : 0.

COROLLARY VI.*

§ 225. If the estate is held on the *longest* of any number of lives (that is, on condition that, whenever *all* those lives become extinct, the lease may be renewed with the same number of lives, and on the same conditions, by paying the given fine) the formulæ in the last corollary, since all the lives may in this case be considered but as one, will still express the true value of the landlord's interest; if we make A denote the value of an annuity on the *longest* of all the lives in possession, and $\overset{1}{A}$ the assumed value of an annuity on the *longest* of all the lives with which the lease is to be continually filled up.

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* Simpson's *Sup.* Prob. 28. De Moivre, Prob. 12. Dodson, vol. iii. Ques. 85.

§ 226. *Example.* Suppose an estate to be leased on two lives, with condition that, on the extinction of both those lives, the same may be renewed with two other lives (the best that can be found) on paying a fine of £300; and so on for ever: what is the present value of the landlord's interest in the estate: taking the probabilities of living as at *Northampton*, and the rate of interest at 5 per cent.?

Here we shall have $\overset{1}{A}$ (or the value of an annuity on the longest of two lives, both aged 8 years) $=$ 17·721, $\rho = \cdot 05$, and $f = 300$; consequently the value of the landlord's interest *immediately before* the receipt of a fine will be equal to $300 \times \frac{1}{\cdot 05 \times 17 \cdot 721} =$ 340·656, or £340 : 13 : 1: and *immediately after* the receipt of a fine it will be equal to 40·656. But, if the ages of the lives, on which the estate is now held, be 40 and 60 years of age, the landlord's interest will be equal to $300 \times \frac{1 - \cdot 05 \times 13 \cdot 214}{\cdot 05 \times 17 \cdot 721} = 114 \cdot 880$: or, if the eldest of those lives be extinct, the landlord's interest will be equal to $300 \times \frac{1 - \cdot 05 \times 11 \cdot 837}{\cdot 05 \times 17 \cdot 721} = 138 \cdot 192$.

SCHOLIUM.

§ 227. From the principles here laid down, it will be easy to determine whether it is most advantageous, to the lessee or the landlord, to fill up a life as soon as it becomes vacant; or to wait till two or more of them have dropt before the renewal.

CHAPTER VIII.

On Assurances depending on a particular Order of Survivorship.

§ 228. THE subject of the present chapter is certainly one of the most intricate in the whole doctrine of annuities, since it involves contingencies for which it is very difficult to give a concise and accurate expression. When two lives only are concerned, the investigations are not very complex; and the solutions may be obtained without much labour or inconvenience: but when three or more lives are involved in the question, the investigations become more intricate, and in many cases indeed baffle all our endeavours to obtain the correct value. These latter cases, which are equally numerous with those whose values we can obtain correctly, arise out of the subject already mentioned in page 123; and will mostly occur towards the end of this chapter: we may, indeed, approximate to their true value by the help of the two lemmata given in the fifth chapter; as will more distinctly appear hereafter.

I would here observe that I have not considered any cases where *more* than three lives are involved: those cases are so very rare that it would not be worth while to lay down any general rules on the subject; and to investigate them properly would swell the present work to an enormous bulk.

In order to avoid any unnecessary repetitions in the ensuing problems, I will take this opportunity of mentioning, once for all, that I shall in every case denote the given sum by s; and that the probabilities of living will be still represented by the same quantities as in § 23. The resulting formulæ, which show the value of such sum, will sufficiently enable the experienced analyst to determine its numerical value: but they are too complex and intricate to be inserted as rules, in words at length.

§ 229. I would also observe here, that I use the characters $\overset{\prime}{A}$, $\overset{\prime}{B}$, $\overset{\prime}{C}$ to denote the value of an annuity on a life one year *older* than the life A, B, or C respectively: and the characters $\underset{\prime}{A}$, $\underset{\prime}{B}$, $\underset{\prime}{C}$ to denote the value of an annuity on a life one year *younger* than the life A, B, or C respectively. Consequently when the character $\overset{\prime}{A}{}^{n}$, $\overset{\prime}{B}{}^{n}$, or $\overset{\prime}{C}{}^{n}$ occurs, it is meant to denote the value of an annuity on a life n years older than $\overset{\prime}{\mathrm{A}}$, $\overset{\prime}{\mathrm{B}}$, or $\overset{\prime}{\mathrm{C}}$ respectively: that is, on a life $(n+1)$ years older than A, B, or C respectively.

The same observations will apply to the characters $\underset{\prime}{A}{}^{n}$, $\underset{\prime}{B}{}^{n}$, or $\underset{\prime}{C}{}^{n}$, which respectively denote the value of an annuity on a life n years older than $\underset{\prime}{\mathrm{A}}$, $\underset{\prime}{\mathrm{B}}$ or $\underset{\prime}{\mathrm{C}}$: that is, on a life $(n-1)$ years older than A, B, or C, respectively. This remark will also extend to the case of such lives considered jointly with any other lives: thus $\overset{\prime}{A}B$ denotes the value of an annuity on the joint lives of B and a life one year older than A; $\overset{\prime}{A}BC$ the value of an annuity on the joint lives of BC and a life one year older than A: and so on.

PROBLEM XXVII.*

§ 230. To determine the present value of a given sum, payable on the decease of A, provided that shall be the *first* which fails of two given lives A, B.

SOLUTION.

The chance of receiving the sum at the end of any one year will depend on the happening of one or other of these two events: 1° that A dies *in* the year, and that B lives to the end of it: 2° that both lives fail *in* the year, restrained however to the contingency that A dies first. The probability that the first event will happen in the first year is $\frac{(a-\overset{\prime}{a})\,\overset{\prime}{b}}{ab}$; and the probability that the second event will happen in the same period is $\frac{(a-\overset{\prime}{a})\times(b-\overset{\prime}{b})}{2ab}$: these two values, therefore, being added together and multiplied by $s(1+\varrho)^{-1}$, or the present value of the given sum *certain* to be received at the end of the year, will give $\frac{s}{2}(1+\varrho)^{-1} \times \left(\frac{ab}{ab} - \frac{\overset{\prime}{a}\overset{\prime}{b}}{ab} - \frac{\overset{\prime}{a}b}{ab} + \frac{a\overset{\prime}{b}}{ab}\right)$ for the value of the expectation of receiving such sum at the end of the first year. In like manner, since the probability that the first event will happen *in* the second year is $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}}{ab}$, and the probability that the second event will happen *in* the same period is $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}) \times (\overset{\prime}{b}-\overset{\prime\prime}{b})}{2ab}$, it follows that the sum of these, multiplied by $s(1+\varrho)^{-2}$, will give the value of the expectation of receiving the sum at the end of the second year. By a similar method of reasoning we may find the value of the expectation of receiving the sum at the end of the third year: and so on for every succeeding year, to the utmost extent of human life; the sum of all which yearly values, or the series

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* Price, Ques. 11, and Note (M). Dodson, vol. iii. Ques. 23. Simpson's *Sup.* Prob. 32. Morgan, Prob. 16, and in Phil. Trans. for 1788, Prob 2.

$$
\begin{array}{l}
\frac{s}{2}(1+\varrho)^{-1} \times \left[\frac{ab}{ab} - \frac{\overset{\prime}{a}\overset{\prime}{b}}{ab} - \frac{\overset{\prime}{a}b}{ab} + \frac{a\overset{\prime}{b}}{ab}\right] + \\
\frac{s}{2}(1+\varrho)^{-2} \times \left[\frac{\overset{\prime}{a}\overset{\prime}{b}}{ab} - \frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{ab} - \frac{\overset{\prime\prime}{a}\overset{\prime}{b}}{ab} + \frac{\overset{\prime}{a}\overset{\prime\prime}{b}}{ab}\right] + \\
\frac{s}{2}(1+\varrho)^{-3} \times \left[\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{ab} - \frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{ab} - \frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime}{b}}{ab} + \frac{\overset{\prime\prime}{a}\overset{\prime\prime\prime}{b}}{ab}\right] + \\
\qquad \&c \qquad\qquad \&c \qquad\qquad \&c,
\end{array}
$$

will be the total present value of the sum s to be received on the above contingency.

§ 231. But, the sum of the first two of these perpendicular series (independent of the common multiplier $\frac{s}{2}$) is by Prob. XXII, equal to $\frac{1-\varrho AB}{(1+\varrho)}$; the third will be found equal to $-\frac{1+\overset{\prime}{A}B}{(1+\varrho)}\times\frac{\overset{\prime}{a}}{a}$; and the last, equal to $\underset{\prime}{A}B\times\frac{\underset{\prime}{a}}{a}$.* Consequently the total present value of the given sum will be equal to $\frac{s}{2}\left[\frac{1-\varrho AB}{(1+\varrho)}-\left(\frac{(1+\overset{\prime}{A}B)\overset{\prime}{a}}{(1+\varrho)}-\underset{\prime}{A}B.\underset{\prime}{a}\right)\frac{1}{a}\right]$.

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* The third of these perpendicular series, or $\frac{\overset{\prime}{a}b}{ab(1+\varrho)}+\frac{\overset{\prime\prime}{a}\overset{\prime}{b}}{ab(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime}{b}}{ab(1+\varrho)^3}$ + &c (by taking all the terms as affirmative and omitting the common multiplier $\frac{s}{2}$), is evidently equal to $\frac{\overset{\prime}{a}b}{ab(1+\varrho)}\left[1+\frac{\overset{\prime\prime}{a}\overset{\prime}{b}}{\overset{\prime}{a}b(1+\varrho)}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime}{b}}{\overset{\prime}{a}b(1+\varrho)^2}+\frac{\overset{\prime\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{ab(1+\varrho)^3}+\&c\right]$: which (since $\frac{\overset{\prime\prime}{a}\overset{\prime}{b}}{\overset{\prime}{a}b(1+\varrho)}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime}{b}}{\overset{\prime}{a}b(1+\varrho)^2}+\frac{\overset{\prime\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{\overset{\prime}{a}b(1+\varrho)^3}$ + &c is equal to $\overset{\prime}{A}B$) will become equal to $\frac{\overset{\prime}{a}b}{ab(1+\varrho)}(1+\overset{\prime}{A}B)=\frac{1+\overset{\prime}{A}B}{(1+\varrho)}\times\frac{\overset{\prime}{a}}{a}$. In like manner, it will be evident that the fourth perpendicular series, or $\frac{a\overset{\prime}{b}}{ab(1+\varrho)}+\frac{\overset{\prime}{a}\overset{\prime\prime}{b}}{ab(1+\varrho)^2}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime\prime}{b}}{ab(1+\varrho)^3}$ + &c, is equal to $\frac{\underset{\prime}{a}}{a}\left[\frac{a\overset{\prime}{b}}{\underset{\prime}{a}b(1+\varrho)}+\frac{\overset{\prime}{a}\overset{\prime\prime}{b}}{\underset{\prime}{a}b(1+\varrho)^2}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime\prime}{b}}{\underset{\prime}{a}b(1+\varrho)^3}+\&c\right]$: which (since $\frac{a\overset{\prime}{b}}{\underset{\prime}{a}b(1+\varrho)}+\frac{\overset{\prime}{a}\overset{\prime\prime}{b}}{\underset{\prime}{a}b(1+\varrho)^2}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime\prime}{b}}{\underset{\prime}{a}b(1+\varrho)^3}$ + &c is equal to $\underset{\prime}{A}B$) will become equal to $\underset{\prime}{A}B\times\frac{\underset{\prime}{a}}{a}$.

The value of these series, however, may be expressed in a different manner, by inverting the method here pursued: for, the third perpendicular series is (on the assumption just mentioned)

As this formula will often be referred to in the subsequent problems, it will be convenient to denote it by a more simple expression; therefore let it be represented by $\mathscr{A}^B$: that is, let $\mathscr{A}^B$ denote the present value of £1 to be received on the contingency mentioned in the problem; consequently $s \times \mathscr{A}^B$ will also equal to $\frac{\underset{\prime}{b}}{b}\left[\frac{\overset{\prime}{a}\,b}{a\underset{\prime}{b}(1+\varrho)} + \frac{\overset{\prime\prime}{a}\,\overset{\prime}{b}}{a\underset{\prime}{b}(1+\varrho)^2} + \frac{\overset{\prime\prime\prime}{a}\,\overset{\prime\prime}{b}}{a\underset{\prime}{b}(1+\varrho)^3} + \&c\right] = \underset{\prime}{A}B \times \frac{\underset{\prime}{b}}{b}$: and the fourth perpendicular series is also equal to $\frac{a\overset{\prime}{b}}{ab(1+\varrho)}\left[1 + \frac{\overset{\prime}{a}\,\overset{\prime\prime}{b}}{\overset{\prime}{a}b(1+\varrho)} + \frac{\overset{\prime\prime}{a}\,\overset{\prime\prime\prime}{b}}{\overset{\prime}{a}b(1+\varrho)^2} + \frac{\overset{\prime\prime\prime}{a}\,\overset{\prime\prime\prime\prime}{b}}{\overset{\prime}{a}b(1+\varrho)^3} + \&c\right] =$ $\frac{a\overset{\prime}{b}}{ab(1+\varrho)}(1+A\overset{\prime}{B}) = \frac{1+A\overset{\prime}{B}}{(1+\varrho)} \times \frac{\overset{\prime}{b}}{b}$. Whence it appears that

$$\frac{1+\overset{\prime}{A}B}{(1+\varrho)} \times \frac{\overset{\prime}{a}}{a} = A\underset{\prime}{B} \times \frac{\underset{\prime}{b}}{b}$$

$$\frac{1+A\overset{\prime}{B}}{(1+\varrho)} \times \frac{\overset{\prime}{b}}{b} = \underset{\prime}{A}B \times \frac{\underset{\prime}{a}}{a}:$$

and consequently, (since *each* series may be summed up in two different ways) that we may adopt either expression for the value of the same: a circumstance of which I have availed myself in some of the subsequent problems. The same observations will apply to any other two lives.

Mr. Morgan (apparently not aware of this property) has run into some strange errors on this subject. He has given *both* expressions in the *same* formula: and such expressions, being there always used with *contrary* signs, of course destroy each other; and are therefore *unnecessarily* introduced! These cases are by no means singular, as they occur in almost *all the problems* (involving similar contingencies) inserted by him in the Philosophical Transactions. See a remarkable instance of this kind in the note to the First Case of Problem XLV.

denote the present value of the given sum under the came circumstances.*

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* The only methods of solving this problem (previous to the investigation of it by Mr. Morgan from the real probabilities of life) were those given by Mr. Simpson in the *Supplement* to his *Doctrine of Annuities,* Prob. XXXII, and by Mr. Dodson in the third volume of his *Mathematical Repository,* Ques. XXIII: both of which are deduced from M. De Moivre's hypothesis. Mr. Simpson however has inserted a rule which may be applied to any table of observations: whilst Mr. Dodson still abides by the hypothesis of his friend De Moivre. Mr. Morgan, with his usual antipathy against that theory, has despatched Mr. Dodson's rule in very few words, by saying that "having derived his rule from a wrong hypothesis, "he has rendered it of *no use.*" (*Phil. Trans.* vol. lxxviii. p. 332): and Dr. Price has observed that he knows of *no other* method of solution but that of Mr. Simpson. Had these gentlemen, however, taken the trouble to compare Mr. Simpson's rule with that given by Mr. Dodson, they would have found that the latter (even on De Moivre's hypothesis) gives the values oftentimes more correctly than the former: and were it adapted to the real probabilities of life, in the same manner as they have adapted Mr. Simpson's, it would in many cases give the values still more corresponding to the *true* values. I agree with them however that neither of these methods can be safely used, except in the middle stages of life: and that it will be best in *all* cases to deduce the values from real observations. Nevertheless as a *near* value may be oftentimes required, without much labour of computation, I have thought it necessary to point out how far these methods may be depended on.

But, is it not singular that, after the unlimited censure which Mr. Morgan has cast upon the "wretched" hypothesis of De Moivre, he should (as editor of Dr. Price's *Observations on Reversionary Payments*) suffer the Fortieth Table in that work to remain without a comment? particularly as he must well know, because he has taken much pains to prove, that the values in that

COROLLARY I.

§ 232. Having thus found the present value of the given sum on the contingency of B surviving A; the present value of the same sum on the contingency of A surviving B (that is, of the same sum to be received on the death of B, provided his life be the fisrt that becomes extinct) is readily found by substituting the symbols a, $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, &c for b, $\overset{\prime}{b}$, $\overset{\prime\prime}{b}$, &c in the above analysis: whence such value will come out equal to

$$\frac{s}{2}\left[\frac{1-\varrho AB}{(1+\varrho)}-\left(\frac{(1+A\overset{\prime}{B})\overset{\prime}{b}}{(1+\varrho)}-A\underset{\prime}{B}.\underset{\prime}{b}\right)\frac{1}{b}\right].$$

table are *extremely erroneous*, and in many cases are more than *one third* of their true value too much. And is it not more singular that, to this very hour, it should serve to determine the value of such assurances at the Equitable Society, the business of which Mr. Morgan has so long and so ably conducted! It is true, that, the concerns of that society being, for the most part, established upon such fair and truly equitable principles, it little signifies *how much* is paid for an assurance provided every one pays *in proportion:* but it must be evident that, in the present case, the rest of the society are benefited at the expense of those who assure on the contingency mentioned in this problem. And it is amusing to observe how blindly the other established Offices, as well as the new ephemeral Companies, have followed this error of their great prototype.

I have inserted the table, above alluded to, at the end of the present work (being the fifty-third table there given); not only as a matter of curiosity, but to show upon what erroneous principles the business of those Offices is sometimes conducted. See the Scholium to Question XXX in Chapter XII.

But, if the present value of the given sum to be received on the death of A according to the problem be once determined, we may easily find the value of the same sum to be received on the death of B, provided his life be the first that fails, by subtracting the value found by the problem from the present value of the given sum to be received on the extinction of the joint lives AB, as found by Prob. XXII: or, in other words, by changing the sign of the second term in the general expression deduced from the problem. Thus $\frac{s}{2}\left[\frac{1-\varrho AB}{(1+\varrho)}+\left(\frac{(1+\overset{\prime}{A}B)\overset{\prime}{a}}{(1+\varrho)}-\underset{\prime}{A}B.\underset{\prime}{a}\right)\frac{1}{a}\right]$ will also denote the present value of the given sum on the contingency of A surviving B.*

As these formulæ will be often referred to in the subsequent problems, it will be more convenient to denote them by a more simple expression; therefore let them be represented by $\boldsymbol{B}^{A}$: that is, let $\boldsymbol{B}^{A}$ denote the present value of £1 to be received on the contingency mentioned in this corollary; consequently $s\times\boldsymbol{B}^{A}$ will denote the present value of the given sum under the same circumstances.

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* Universally we shall find that, by this method of investigation, the expression $\frac{s}{2}\left[\frac{1-\varrho AB}{(1+\varrho)}\mp\left(\frac{(1+\overset{\prime}{A}B)\overset{\prime}{a}}{(1+\varrho)}-\underset{\prime}{A}B.\underset{\prime}{a}\right)\frac{1}{a}\right]=$ $\frac{s}{2}\left[\frac{1-\varrho AB}{(1+\varrho)}\pm\left(\frac{(1+A\overset{\prime}{B})\overset{\prime}{b}}{(1+\varrho)}-\underset{\prime}{A}B.\underset{\prime}{b}\right)\frac{1}{b}\right]$ will denote the value in either case; where the upper sign is to be taken if the sum depends on the contingency of A's dying first; and the lower sign, if on B's dying first.

COROLLARY II.

§ 233. From the principles laid down in the problem we may readily find the value of any sum depending on the contingency of C surviving A; that is, of such sum to be received on the death of A provided his life be the first that becomes extinct of the two lives A, C. The value of such sum will be equal to $\frac{s}{2}\left[\frac{1-\varrho AC}{(1+\varrho)}-\left(\frac{(1+\overset{\prime}{A}C)\overset{\prime}{a}}{(1+\varrho)}-\underset{\prime}{A}C.\underset{\prime}{a}\right)\frac{1}{a}\right]$: and which I shall (for the reasons above mentioned) denote by $s\times\mathcal{A}^{C}$.

In like manner the value of any sum depending on the contingency of C surviving B will be found equal to $\frac{s}{2}\left[\frac{1-\varrho BC}{(1+\varrho)}-\left(\frac{(1-\overset{\prime}{B}C)\overset{\prime}{b}}{(1+\varrho)}-\underset{\prime}{B}C.\underset{\prime}{b}\right)\frac{1}{b}\right]$: which I shall for the same reasons denote by $s\times\mathcal{B}^{C}$. These formulæ will often be referred to in the subsequent problems.

COROLLARY III.

§ 234. If the contingency mentioned in the problem, and on which the given sum is to be received, is *Deferred* for any given number of years ($=n$), less than that to which it is probable the two joint lives may extend, it is evident that the several perpendicular series, in the general expression given in page 184, must be continued from after the n^{th} term only; and thence to the utmost extent of human life. The sum of which terms will, by an attention to the steps pursued in Problem XXII, cor. 1, be found equal to $\frac{s}{2}\left[\frac{1-\varrho A^{\prime}B^{\prime}}{(1+\varrho)}\times\frac{\alpha\beta}{ab}(1+\varrho)^{-n}-\frac{1+\overset{\prime}{A}{}^{\prime}B^{\prime}}{(1+\varrho)}\times\frac{\overset{\prime}{\alpha}\beta}{ab}(1+\varrho)^{-n}+\underset{\prime}{A}{}^{\prime}B^{\prime}\times\frac{\underset{\prime}{\alpha}\beta}{ab}(1+\varrho)^{-n}\right]=\frac{s\beta}{2ab}(1+\varrho)^{-(n+1)}\times\left[(1-\varrho A^{\prime}B^{\prime})\alpha-(1+\overset{\prime}{A}{}^{\prime}B^{\prime})\overset{\prime}{\alpha}+\underset{\prime}{A}{}^{\prime}B^{\prime}(1+\varrho)\underset{\prime}{\alpha}\right]$:* an expression which I shall denote by $s\times(\mathcal{A}^{B})^{\mathrm{d}}$.

COROLLARY IV.

§ 235. Or, if the contingency, on which the given sum is to be received, is *Temporary* (that is, to continue only for a given *term* = n, less than that to which it is probable the two joint lives may extend) it is evident that the several perpendicular series, in the general expression given in page 184, must be continued for n terms only: and the sum of these will, by an attention to the steps pursued in Prob. XXII, cor. 2, be found equal to $\frac{s}{2}\left[\frac{1+(AB)^{\mathrm{t}-1}}{(1+\varrho)}-(AB)^{\mathrm{t}}-\left(\frac{1+(\overset{\prime}{A}B)^{\mathrm{t}-1}}{(1+\varrho)}.\overset{\prime}{a}-(\underset{\prime}{A}B)^{\mathrm{t}}.\underset{\prime}{a}\right)\frac{1}{a}\right]$.

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* It must be remembered that $\overset{\prime}{A}{}^{\prime}B^{\prime}$ denotes the value of an annuity on two joint lives, each n years older than $\overset{\prime}{A}$ and B; and that $\underset{\prime}{A}{}^{\prime}B^{\prime}$ denotes the value of an annuity on two joint lives, each n years older than $\underset{\prime}{A}$ and B respectively: agreeably to what has been said in § 220.

But, since $(\overset{\prime}{A}B)^{t-1} + \frac{\overset{\prime}{\alpha}\beta}{\overset{\prime}{a}b(1+\varrho)^{n}} = (\overset{\prime}{A}B)^{t} = \overset{\prime}{A}B - \overset{\prime}{A}B' \times \frac{\overset{\prime}{\alpha}\beta}{\overset{\prime}{a}b}(1+\varrho)^{-n}$, we may (by substituting these values, as explained in the problem just referred to) render this formula more convenient for practice by the following expression $\frac{s}{2}\left[\frac{1-\varrho AB}{(1+\varrho)} - \frac{1-\varrho A'B'}{(1+\varrho)} \times \frac{\alpha\beta}{ab}(1+\varrho)^{-n} - \frac{1+\overset{\prime}{A}B}{(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a} + \frac{1+\overset{\prime}{A}'B'}{(1+\varrho)}\cdot\frac{\overset{\prime}{\alpha}\beta}{ab}(1+\varrho)^{-n} + \underset{\prime}{A}B\cdot\frac{\underset{\prime}{a}}{a} - \underset{\prime}{A}'B'\cdot\frac{\underset{\prime}{\alpha}\beta}{ab}(1+\varrho)^{-n}\right] = s.\mathcal{A}^{B} - s.(\mathcal{A}^{B})^{d}$: that is, equal to the difference between the present value of the sum found by the problem, and the present value of the same sum *Deferred* as in the preceding corollary.*

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* I believe that this formula is the first *correct* rule that has ever been given for determining the value of such assurances for given *terms*. In the solution of the fifteenth question in Dr. Price's *Obs: on Rev. Pay.* the author has given a rule by which we may obtain a *near* value of such assurances, and has referred to note H for a *demonstration* of the same. This demonstration (as he calls it) would indeed lead us to the *exact* value according to any table of observations, agreeably to what the author has justly remarked: the *rule*, however, has not been deduced from this investigation, but from Question XXIII in the third volume of Mr. Dodson's *Mathematical Repository*, or from the process laid down by Mr. Simpson in the *Supplement* to his Doctrine of Annuities, page 62.

Mr. Morgan, by way of improving on this *explanation*, has given a rule (without a demonstration) which he says is obtained "without the aid of M. De Moivre's hypothesis." This, however, is not strictly true; for, his rule is deduced on the supposition that

COROLLARY V.

§ 236. If the two lives are equal to each other, or of the same age A, the general expression in the problem becomes equal to $\frac{s}{2}\times\frac{1-\varrho AA}{(1+\varrho)}$; because the second term in such expression, or $\frac{1}{a}\left(\frac{(1+\overset{\prime}{A}A)\overset{\prime}{a}}{(1+\varrho)}-\underset{\prime}{A}A.\underset{\prime}{a}\right)$, will then vanish altogether: whence it follows that the value will in such case be equal to one

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the decrements of life are equal during the given term; similar to the method explained in the note in page 150 of the present work: and this use of M. De Moivre's hypothesis but little accords with what he has repeatedly advanced, "that it is best to exclude it *entirely* from the doctrine of survivorship." This very fact will show that it may occasionally be used with good effect, even in the solution of the very problem where he has censured it the most. I shall here subjoin an investigation of Mr. Morgan's rule, which will show the method of proceeding in similar cases, when we wish to adopt the same principles.

It will appear that the general series in the problem may be converted into the following one,

$$\frac{s}{2(1+\varrho)}\times\frac{a-\overset{\prime}{a}}{a}\left(\frac{b}{b}+\frac{\overset{\prime}{b}}{b}\right)+$$

$$\frac{s}{2(1+\varrho)^2}\times\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{a}\left(\frac{\overset{\prime}{b}}{b}+\frac{\overset{\prime\prime}{b}}{b}\right)+$$

$$\frac{s}{2(1+\varrho)^3}\times\frac{\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a}}{a}\left(\frac{\overset{\prime\prime}{b}}{b}+\frac{\overset{\prime\prime\prime}{b}}{b}\right)+$$

$$\&c \qquad\qquad \&c.$$

Now, if we suppose the decrements of A's life to be equal during the given term, (which in many cases will be found to correspond very nearly with the fact,) and if such decrements be de-half the present value of the given sum to be received on the extinction of the joint lives.

This observation will equally apply to the formulæ, in the last two corollaries, for *terms*; and is indeed self-evident. For, as the given sum is not to be re-

noted by $\delta = \frac{a-\alpha}{n}$; then will the first n terms of this series be equal to

$$\frac{s.\delta}{2a(1+\varrho)}\left(\frac{b}{b}+\frac{\overset{\prime}{b}}{b}\right)+$$
$$\frac{s.\delta}{2a(1+\varrho)^2}\left(\frac{\overset{\prime}{b}}{b}+\frac{\overset{\prime\prime}{b}}{b}\right)+$$
$$\frac{s.\delta}{2a(1+\varrho)^3}\left(\frac{\overset{\prime\prime}{b}}{b}+\frac{\overset{\prime\prime\prime}{b}}{b}\right)+$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\frac{s.\delta}{2a(1+\varrho)^n}\left(\frac{\underset{\prime}{\beta}}{b}+\frac{\beta}{b}\right).$$

But, the first of these perpendicular series, within the parentheses, is (independent of the common multiple $\frac{s.\delta}{2a}$) equal to $\frac{1+(B)^{t-1}}{(1+\varrho)}$; and the second is, in like manner, equal to $(B)^t$: whence the total present value of the assurance for the given term, will upon this hypothesis be denoted by $\frac{s.\delta}{2a}\left[\frac{1+(B)^{t-1}}{(1+\varrho)}+(B)^t\right] = \frac{s(a-\alpha)}{2an}\left[\frac{1+(B)^{t-1}}{(1+\varrho)}+(B)^t\right]$: and whence Mr. Morgan's rule is manifest.

But this rule may be more conveniently expressed by $\frac{s(a-\alpha)}{2an} \times \frac{1+(2+\varrho)B-[1+(2+\varrho)B^2]\times\frac{\beta}{b}(1+\varrho)^{-n}}{(1+\varrho)}$: which, when the two series are continued to the utmost extent of human life (B being the oldest), will become $\frac{s(a-\alpha)}{2an} \times \frac{1+(2+\varrho)B}{1+\varrho}$. In this latceived till after the extinction of the joint lives, and as it is certainly an *even* chance which of the two may survive, it is manifest that one *half* the present value of the sum to be received on the extinction of those joint lives will be the true value of the sum depending on this contingency; whether such contingency extends to the whole continuance of the given lives, or exists only for a given term.

Observations on Mr. Morgan's Method of investigating this Problem.

§ 237. I cannot dismiss this problem without some remarks on the singular and unscientific manner in which Mr. Morgan has summed up the four collateral series (in page 184) which express the present value of the given sum to be received on the contingency mentioned in the problem: and I am the more induced to do this because similar expressions arise in many of his subsequent problems, in which the same loose and eccentric method has been pursued; whereby an air of obscurity and difficulty has been thrown over the subject, which it is my intention here

ter case, however, the formula cannot be depended upon, unless A is much the youngest of the two lives; but at the same time not younger than 15 years of age: indeed it must at *all* times be used with proper caution, and with a due regard to the tables of observations made use of. These remarks will equally apply to the solution of the sixteenth question in Dr. Price's work above quoted.

to remove. Mr. Morgan divides the whole expression into the two following collateral series,* viz.

$$\frac{s}{2(1+\varrho)}\left[\frac{(a-a')b'}{ab}+\frac{(a-a')b}{ab}\right]+$$
$$\frac{s}{2(1+\varrho)^2}\left[\frac{(a'-a'')b''}{ab}+\frac{(a'-a'')b'}{ab}\right]+$$
$$\frac{s}{2(1+\varrho)^3}\left[\frac{(a''-a''')b'''}{ab}+\frac{(a''-a''')b''}{ab}\right]+$$
$$\&c \qquad\qquad \&c;$$

which is evidently the same series as that which I have given in page 184.

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* In the subsequent process given in the text I have adopted the method of notation made use of throughout the whole of this work, in order to prevent any obscurity or circumlocution by the introduction of new symbols. But, lest I should be charged with having misrepresented Mr. Morgan, I shall here subjoin the method which he himself has pursued in investigating this problem; and for the sake of greater accuracy shall adopt the same characters which he has used.

Let r denote £1 increased by its interest for a year: let a represent the number of persons living in the table at the age of A; and let a', a'', a''', &c, represent the decrements at the end of the first, second, third, &c, years from the age of A. Let b represent the number of persons living at the age of B; and let c, d, e, f, &c, represent the number of persons living at the end of the first, second, third, fourth, &c, years from the age of B.

Then (in the words of Mr. Morgan, in *Phil. Trans.* for 1788, p. 342) it will follow that the present value of S to be received on the death of A, should that happen in the life-time of B, will be expressed by the series $S \times \left[\frac{(b+c)a'}{2abr}+\frac{(c+d)a''}{2abr^2}+\frac{(d+e)a'''}{2abr^3}+\frac{(e+f)a''''}{2abr^4}+\&c\right]$. This series may be resolved into the two fol-

§ 238. The first of these collateral series he expands into a more complex one, by making it equal to

$$\frac{s}{2(1+\varrho)}\left[\frac{\overset{\prime}{b}}{b}-\frac{\overset{\prime}{a}\overset{\prime}{b}}{ab}\;.\;.\;.\;.\;.\;\right]+$$
$$\frac{s}{2(1+\varrho)^2}\left[\frac{\overset{\prime\prime}{b}}{b}-\frac{\overset{\prime\prime}{a}\overset{\prime\prime}{b}}{ab}-\frac{\overset{\prime\prime}{b}}{b}+\frac{\overset{\prime}{a}\overset{\prime\prime}{b}}{ab}\right]+$$
$$\frac{s}{2(1+\varrho)^3}\left[\frac{\overset{\prime\prime\prime}{b}}{b}-\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}}{ab}-\frac{\overset{\prime\prime\prime}{b}}{b}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime\prime}{b}}{ab}\right]+$$
$$\&c \qquad\qquad \&c \qquad\qquad \&c;$$

and he then proceeds to sum up *these* several collateral series in the following manner. The first he makes (independent of the common multiple $\frac{s}{2}$) equal to B, the second equal to $-AB$, the third equal to $-\frac{\overset{\prime}{B}}{(1+\varrho)}\times\frac{\overset{\prime}{b}}{b}$, and the fourth equal to $\frac{A\overset{\prime}{B}}{(1+\rho)}\times\frac{\overset{\prime}{b}}{b}$:

lowing; $\frac{S}{2}\times\left[\frac{c\overset{\prime}{a}}{abr}+\frac{d\overset{\prime\prime}{a}}{abr^2}+\frac{e\overset{\prime\prime\prime}{a}}{abr^3}+\frac{f\overset{\prime\prime\prime\prime}{a}}{abr^4}+\&c\right]+\frac{S}{2}\times\left[\frac{b\overset{\prime}{a}}{abr}+\frac{c\overset{\prime\prime}{a}}{abr^2}+\frac{d\overset{\prime\prime\prime}{a}}{abr^3}+\frac{e\overset{\prime\prime\prime\prime}{a}}{abr^4}+\&c\right]$. The first of these two may be again resolved into $\frac{S}{2}\times\left[\frac{c}{br}-\frac{ca-c\overset{\prime}{a}}{abr}+\frac{d}{br^2}-\frac{da-d\overset{\prime}{a}-d\overset{\prime\prime}{a}}{abr^2}+\frac{e}{br^3}-\frac{ea-e\overset{\prime}{a}-e\overset{\prime\prime}{a}-e\overset{\prime\prime\prime}{a}}{abr^3}+\&c\right]-\frac{Sc}{2br}\times\left[\frac{d}{cr}-\frac{da-d\overset{\prime}{a}}{acr}+\frac{e}{cr^2}-\frac{ea-e\overset{\prime}{a}-e\overset{\prime\prime}{a}}{acr^2}+\&c\right]$. Let B denote the value of an annuity on the life of B, C the value of an annuity on a life one year older than B, AB and AC the values of annuities on the joint lives of A and B and of A and C, and these series will be $=\frac{S\times(B-AB)}{2}-\frac{S\times c(C-AC)}{2br}$. Again, the second series above mentioned, or $\frac{S}{2}\times\left[\frac{b\overset{\prime}{a}}{abr}+\frac{c\overset{\prime\prime}{a}}{abr^2}+\frac{d\overset{\prime\prime\prime}{a}}{abr^3}+\&c\right]$, by so that the total value of these four collateral series (or of the *first* collateral series in page 196) is made equal to $B - \frac{\overset{\prime}{B}}{(1+\varrho)} \times \frac{\overset{\prime}{b}}{b} + \frac{A\overset{\prime}{B}}{(1+\varrho)} \times \frac{\overset{\prime}{b}}{b} - AB$.

And, by proceeding in a similar manner with respect to the *second* collateral series in page 196, the sum of it is found equal to $\underset{\prime}{B} \times \frac{\underset{\prime}{b}}{b} - \frac{B}{(1+\varrho)} - A\underset{\prime}{B} \times \frac{\underset{\prime}{b}}{b} + \frac{AB}{(1+\varrho)}$. Consequently, the total present value of the given sum, to be received on the contingency mentioned in the problem, is equal to $\frac{s}{2}\left[\underset{\prime}{B} \times \frac{\underset{\prime}{b}}{b} - \frac{B}{(1+\varrho)} + \frac{AB}{(1+\varrho)} - AB + B - \frac{\overset{\prime}{B}}{(1+\varrho)} \times \frac{\overset{\prime}{b}}{b} + \frac{A\overset{\prime}{B}}{(1+\varrho)} \times \frac{\overset{\prime}{b}}{b} - A\underset{\prime}{B} \times \frac{\underset{\prime}{b}}{b}\right]$.

pursuing the same steps may be found $= \frac{\beta S}{2b}(K - AK) - \frac{S \times (B - AB)}{2r}$, where β denotes the number of persons living at the age of a person one year younger than B, K the value of an annuity on that life, and AK the value of an annuity on the joint lives of A and K. The whole value of the survivorship is therefore $= S \times \left[\frac{(r-1) \times (B - AB)}{2r} + \frac{\beta(K - AK)}{2b} - \frac{c(C - AC)}{2br}\right]$.

These are Mr. Morgan's own words: and if the several values, above alluded to, be expressed by the characters which I have adopted in the present work, they will be found to agree with the process given in the text. But $B - \frac{c.C}{br} = \frac{c}{br}$, and $\frac{\beta.K}{b} - \frac{B}{r} = \frac{1}{r}$; whence the absurdity of his formula. Mr. Morgan however has at length altered his *formula* for *this* particular case; although he still preserves the same *method of solution*, both here and in all his subsequent problems. See Price's *Obs.* edit. 1803, note (O).

§ 239. But, surely it cannot be necessary to tell Mr. Morgan that $B - \frac{B'}{(1+\varrho)} \times \frac{b'}{b}$ is equal to $\frac{1}{(1+\varrho)} \times \frac{b'}{b}$;* and that $B_{\prime} \times \frac{b_{\prime}}{b} - \frac{B}{(1+\varrho)}$ is equal to $\frac{1}{(1+\varrho)}$ only!!! Whereby the complex and troublesome formula, above given, may be more simply expressed by $\frac{s}{2}\left[\frac{1-\varrho AB}{(1+\varrho)} + \frac{1+AB'}{(1+\varrho)} \times \frac{b'}{b} - AB_{\prime} \times \frac{b_{\prime}}{b}\right]$, and which is one of the formulæ which I have given, in the note in page 189, for the value of the sum depending on this contingency. Indeed it is difficult to conceive why he should ever have thought it necessary to employ those quantities which denote the value of an annuity on a *single* life; since it is clear from the nature of the several series in page 196, that such values could not properly belong to the subject; and that no quantities are involved in the solution but such as arise from *two joint* lives.

As this strange error runs through the whole of Mr. Morgan's papers in the Philosophical Transactions, where this and similar contingencies are involved, I have thought it right to take this public notice of it;

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* This is indeed evident on inspection of the first and third of the collateral series in page 197. For, it is manifest that all the subsequent terms of those two series *after the first term*, destroy each other; and consequently that the sum of those two collateral series will be equal to the first term only: that is, (independent of the common multiple $\frac{s}{2}$) equal to $\frac{1}{(1+\varrho)} \times \frac{b'}{b}$. In like manner we may prove the equality of the other two quantities alluded to in the text.

not only with a view to obviate any difficulty that may arise from a comparison of his formulæ with those which I have here deduced, but to clear the subject from all diffuse and unconnected matter. And I shall have occasion to make a similar remark on his method of summing the several series where *three* lives are concerned: for which, see the *Observations* in § 251 &c, at the end of Problem XXIX; and in § 272 at the end of Problem XXXV.

PROBLEM XXVIII.*

§ 240. To determine the present value of a given sum payable on the decease of A, provided that shall be the *second* which fails of two given lives A, B.

SOLUTION.

It is evident, in this case, that the chance of receiving the sum at the end of the first year will depend only on one contingency; viz. that both the lives become extinct in that period, restrained however to the condition that A dies last, the probability of which in the first year is $\frac{(a-a')\times(b-b')}{2ab}$; and which, being multiplied by $s(1+\varrho)^{-1}$, or the present value of the given sum, certain to be received at the end of the year, will give $\frac{s}{2}(1+\varrho)^{-1} \times \left(\frac{ab}{ab} + \frac{a'b'}{ab} - \frac{a'b}{ab} - \frac{ab'}{ab}\right)$ for the expectation of receiving such sum at the end of the first year.

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* Price, Ques. 12, and Note (M). Dodson, vol. iii. Ques. 24. Simpson, Prob. 28; and *Sup.* Prob. 33; also Prob. 3 in page 72. De Moivre, Prob. 17. Morgan, Prob. 17; and in Phil. Trans. for 1788, Prob. 3, page 347.

In the second and following years, however, the chance of receiving the sum will depend on the happening of either of two events: 1° that both the lives fail in the year, A having died last: 2° that only A dies in the year, and that B dies in either of the preceding years. The probability that the first event will happen in the second year is $\frac{(a'-a'') \times (b'-b'')}{2ab}$; and the probability of the second event happening in the same period is $\frac{a'-a''}{a}\left(1-\frac{b'}{b}\right)$. These two values therefore, being added together and multiplied by $s(1+\varrho)^{-2}$, will give $\frac{s}{2}(1+\varrho)^{-2} \times \left(\frac{2a'}{a} - \frac{2a''}{a} - \frac{a'b'}{ab} + \frac{a'b''}{ab} + \frac{a''b'}{ab} - \frac{a'b''}{ab}\right)$ for the expectation of receiving the given sum at the end of the second year.

In like manner, it will be found that the probability of the first event happening in the third year is $\frac{(a''-a''') \times (b''-b''')}{2ab}$; and the probability that the second event will happen in the same period is $\frac{a''-a'''}{a}\left(1-\frac{b''}{b}\right)$; therefore these two values being added together and multiplied by $s(1+\varrho)^{-3}$, will give $\frac{s}{2}(1+\varrho)^{-3} \times \left(\frac{2a''}{a} - \frac{2a'''}{a} - \frac{a''b''}{ab} + \frac{a'''b'''}{ab} + \frac{a'''b''}{ab} - \frac{a''b'''}{ab}\right)$ for the expectation of receiving the sum at the end of the third year: and so on for every subsequent year, to the utmost extent of human life; the sum of all which yearly values, or the series*

$$\frac{s}{2}(1+\varrho)^{-1}\times\left[\frac{2a}{a}-\frac{2a'}{a}-\frac{ab}{ab}+\frac{a'b'}{ab}+\frac{a'b}{ab}-\frac{ab'}{ab}\right]+$$
$$\frac{s}{2}(1+\varrho)^{-2}\times\left[\frac{2a'}{a}-\frac{2a''}{a}-\frac{a'b'}{ab}+\frac{a''b''}{ab}+\frac{a''b'}{ab}-\frac{a'b''}{ab}\right]+$$
$$\frac{s}{2}(1+\varrho)^{-3}\times\left[\frac{2a''}{a}-\frac{2a'''}{a}-\frac{a''b''}{ab}+\frac{a'''b'''}{ab}+\frac{a'''b''}{ab}-\frac{a''b'''}{ab}\right]+$$
$$\&c \qquad \&c \qquad \&c,$$

will be the total present value of the sum s to be received on the above contingency.

§ 241. But the sum of the first two of these perpendicular series is, by Prob. XXII, equal to the present value of the given sum to be received on the decease of A; that is, equal to $s\times\frac{1-\varrho A}{(1+\varrho)}$: and the remaining four of these perpendicular series are the same as those produced in the last problem, with a contrary sign; and the sum of which is there denoted by

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* The value of the first year's expectation has been found in the preceding part of the investigation, equal to $\frac{s}{2}(1+\varrho)^{-1}\times\left(\frac{ab}{ab}+\frac{a'b'}{ab}-\frac{a'b}{ab}-\frac{ab'}{ab}\right)$; an expression which is evidently equal to $\frac{s}{2}(1+\varrho)^{-1}\times\left(\frac{2a}{a}-\frac{2a'}{a}-\frac{ab}{ab}+\frac{a'b'}{ab}+\frac{a'b}{ab}-\frac{ab'}{ab}\right)$. This latter value has been assumed in order to correspond with the terms of the subsequent expectations, and thereby render the several collateral series complete. A similar method is pursued in many of the following problems.

$\frac{s}{2}\left[\frac{1-\varrho AB}{(1+\varrho)}-\left(\frac{(1+\acute{A}B)\acute{a}}{(1+\varrho)}-\underset{\prime}{A}B.\underset{\prime}{a}\right)\frac{1}{a}\right]$. Consequently the total value of these several series will be expressed by $s\left[\frac{1-\varrho A}{(1+\varrho)}-A^{B}\right]$.

COROLLARY I.

§ 242. Having thus found the present value of the given sum on the contingency of A dying after B; the present value of the same sum on the contingency of B dying after A (that is, of the same sum to be received on the death of B, provided his life is the second that fails) is readily determined by subtracting the value above found from the present value of the given sum to be received on the extinction of the longest of the two lives; which, by Prob. XXII, cor. 3, is expressed by $s\times\frac{1-\varrho(A+B-AB)}{(1+\varrho)}$. Consequently, the present value of the given sum, on the contingency of B dying after A, will be equal to $\frac{s}{2}\left[\frac{1-\varrho(2B-AB)}{(1+\varrho)}-\left(\frac{(1+\acute{A}\dot{B})\acute{a}}{(1+\varrho)}-\underset{\prime}{A}B\,\underset{\prime}{a}\right)\frac{1}{a}\right]=$ $s\left[\frac{1-\varrho B}{(1+\varrho)}-B^{A}\right]$.

COROLLARY II.

§ 243. If the contingency on which the sum is to be received is *Deferred* for a given number of years $(=n)$, less than that to which it is probable the life A may extend, it is evident that the sum of all the terms of the first two perpendicular series, after the n^{th} year, will, by Prob. XXII, cor. 1, be equal to $s\times\frac{1-\varrho A^{n}}{(1+\varrho)}\times\frac{\alpha}{a}(1+\varrho)^{-n}$; and the sum of all the similar terms of the remaining four perpendicular series will be denoted by the same expression as that which has been deduced in the second corollary to the preceding problem, but with a contrary sign. Therefore if the value found by that corollary be subtracted from the present value of the assurance of the given sum after the extinction of the life A, provided that happens after the given term, the difference will be the value required.

COROLLARY III.

§ 244. If the contingency on which this sum is to be received commences immediately, but continues only for a certain number of years ($= n$), less than that to which it is probable the life A may extend: or, in other words, if we wish to determine the value of a *Temporary* assurance of such sum; it is evident that the several perpendicular series given in the problem, must be continued only for n terms. Now, the sum of the first n terms of the first two of these perpendicular series will, by Prob. XXII, cor. 2, be found equal to $\frac{1-\varrho A}{(1+\varrho)} - \frac{1-\varrho A^{\circ}}{(1+\varrho)} \times \frac{\alpha}{a}(1+\varrho)^{-n}$: from which, if we subtract the first n terms* of the four remaining series, as found by cor. 4 in the preceding problem, the difference will be the value of the assurance for the given term.*

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* When n is *equal* to, or *greater* than, the number of years between the age of B and that age in the table of observations at which human life becomes extinct, we must subtract the *whole* value of the four remaining series as found by the last problem. That is, the correct value in both such cases will be denoted by ϱ multiplied into $\frac{1-\varrho A-(1-\varrho A^{\circ})\frac{\alpha}{a}(1+\varrho)^{-n}}{(1+\varrho)} - \mathcal{A}^{B}$.

* See the note to Cor. 4 in the last problem; where the process there pursued will enable us to find a *near* value of an assurance for a given *term*, according to the conditions of the present problem. For, if we deduct the near value, there deduced, from the near value of an assurance of the same sum on the life of A for the same term, as found by the process laid down in the note to Prob. XXII, cor. 2, the difference, or $\frac{s.\delta}{2a}\left[2 \times \frac{1-(1+\varrho)^{-n}}{\varrho} - \frac{1+(B)^{t-1}}{(1+\varrho)} - (B)^{t}\right]$, will be the value required.

This formula is the same, in fact, as that given by Mr. Morgan in Note H of Dr. Price's *Obs. on Rev. Pay.*; and is another instance of the utility of De Moivre's hypothesis: but, it may be rendered more convenient for practice by the substitutions adopted in the note to Cor. 4 of the preceding problem; whereby it will become $\frac{s(a-\alpha)}{2an}\left[2 \times \frac{1-(1+\varrho)^{-n}}{\varrho} - \frac{1+(2+\varrho)B-[1+(2+\varrho)B] \times \frac{\beta}{b}(1+\varrho)^{-n}}{(1+\varrho)}\right]$.

When n is equal to, or greater than, the number of years between the age of B and that age in the table of observations when human life becomes extinct, β is equal to 0; and consequently $[1+(2+\varrho)B^{\circ}] \times \frac{\beta}{b}(1+\varrho)^{-n}$ vanishes: whereby the formula in both these cases would become $\frac{s(a-\alpha)}{2an}\left[2 \times \frac{1-(1+\varrho)^{-n}}{\varrho} - \frac{1+(2+\varrho)B}{(1+\varrho)}\right]$. But in all *such* cases the value is best obtained by the rule given in the preceding note: since it is then equal to the difference between the value of an assurance on the life A for the given *term*, and the value of an assurance on the *whole* life A provided he be the first that fails.

COROLLARY IV.

§ 245. If the two lives are equal, or of the same age A, the general expression in the problem becomes (agreeably to what has been already said in cor. 5 of the last problem) equal to $\frac{s}{2} \times \frac{1-\varrho(2A-AA)}{(1+\varrho)}$: that is, equal to one half the present value of the given sum to be received on the extinction of the longest of the two lives.

PROBLEM XXIX.*

§ 246. To determine the present value of a given sum payable on the decease of A, provided he be the *first* that fails of three given lives A, B, C.

SOLUTION.

In order to receive the given sum at the end of any one year, it is necessary that one or other of four different events should happen: 1° that all the three lives fail in the year, A having died first: 2° that A and B fail in the year, A having died first, and that C lives to the end of it: 3° that A and C fail in the year, A having died first, and that B lives to the end of it: 4° that only A dies in the year, and that B and C both live to the end of it. The probabilities of the happening of these several events in the first year are respectively $\frac{(a-a').(b-b').(c-c')}{3abc}$, $\frac{(a-a').(b-b')c'}{2abc}$, $\frac{(a-a').(c-c')b'}{2abc}$, and $\frac{(a-a')b'c'}{abc}$; which being added together and multiplied by $s(1+\rho)^{-1}$, will give $\frac{s}{6}(1+\rho)^{-1} \times \left(\frac{2abc}{abc} - \frac{2a'b'c'}{abc} - \frac{2a'bc}{abc} + \frac{2ab'c'}{abc} + \frac{ab'c}{abc} - \frac{a'bc'}{abc} + \frac{abc'}{abc} - \frac{a'b'c}{abc}\right)$ for the expectation of receiving the sum at the end of the first year.

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* Dodson, vol. iii. Ques. 32. Morgan, Prob. 19; and in Phil. Trans. for 1791, Prob. 1, page 248.

In like manner, it will be found that the probabilities of the happening of these several events in the second year are respectively $\frac{(a'-a'').(b'-b'').(c'-c'')}{3abc}$, $\frac{(a'-a'').(b'-b'')c''}{2abc}$, $\frac{(a'-a'').(c'-c'')b''}{2abc}$, and $\frac{(a'-a'')b''c''}{abc}$: which being added together, and multiplied by $s(1+\rho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. And by a similar process we may find the expectation of receiving the sum at the end of the third and every succeeding year, to the utmost extent of human life; the sum of all which yearly values, or the series

$$\frac{}{-\rho)}\left[\frac{2abc}{abc} - \frac{2a'b'c'}{abc} - \frac{2a'bc}{abc} + \frac{2ab'c'}{abc} + \frac{ab'c}{abc} - \frac{a'bc'}{abc} + \frac{abc'}{abc} - \frac{a'b'c}{abc}\right] +$$

$$\frac{}{+\rho)^2}\left[\frac{2a'b'c'}{abc} - \frac{2a''b''c''}{abc} - \frac{2a''b'c'}{abc} + \frac{2a'b''c''}{abc} + \frac{a'b''c'}{abc} - \frac{a''b'c''}{abc} + \frac{a'b'c''}{abc} - \frac{a''b''c'}{abc}\right] +$$

$$\frac{}{+\rho)^3}\left[\frac{2a''b''c''}{abc} - \frac{2a'''b'''c'''}{abc} - \frac{2a'''b''c''}{abc} + \frac{2a''b'''c'''}{abc} + \frac{a''b'''c''}{abc} - \frac{a'''b''c'''}{abc} + \frac{a''b''c'''}{abc} - \frac{a'''b'''c''}{abc}\right] +$$

&c &c &c &c &c,

will be the total present value of the given sum to be received on the contingency above mentioned.

§ 247. But the sums of these eight perpendicular and collateral series (independent of the common multiple s) are respectively equal to $\frac{1+ABC}{3(1+\varrho)} - \frac{ABC}{3} - \frac{1+\overset{\prime}{A}BC}{3(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a} + \underset{\prime}{A}BC\cdot\frac{\underset{\prime}{a}}{3a} + \frac{1+A\overset{\prime}{B}C}{6(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{b} - A\underset{\prime}{B}C\cdot\frac{\underset{\prime}{b}}{6b} + \frac{1+AB\overset{\prime}{C}}{6(1+\varrho)}\cdot\frac{\overset{\prime}{c}}{c} - AB\underset{\prime}{C}\cdot\frac{\underset{\prime}{c}}{6c}$.* Whence, the sum of the first and second is (as in Prob. XXII) equal to $\frac{s}{3}\times\frac{1-\varrho ABC}{(1+\varrho)}$; the sum of the third and fourth is equal to $-\frac{s}{3a}\left[\frac{(1+\overset{\prime}{A}BC)\overset{\prime}{a}}{(1+\varrho)}-\underset{\prime}{A}BC\cdot\underset{\prime}{a}\right]$; the sum of the fifth

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* The sum of the first and second of these collateral series is found by the process laid down in Prob. XXII; and the remaining ones, in the following manner. The sum of the third perpendicular series, or $\frac{\overset{\prime}{a}bc}{abc(1+\varrho)} + \frac{\overset{\prime\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc(1+\varrho)^2} + \frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc(1+\varrho)^3}$ + &c (by taking all the terms as affirmative and leaving out the common multiple $\frac{2s}{6}$) is evidently equal to $\frac{\overset{\prime}{a}bc}{abc(1+\varrho)}\left[1 + \frac{\overset{\prime\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{\overset{\prime}{a}bc(1+\varrho)} + \frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{\overset{\prime}{a}bc(1+\varrho)^2} + \frac{\overset{\prime\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{\overset{\prime}{a}bc(1+\varrho)^3} + \&c\right]$: which, since $\frac{\overset{\prime\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{\overset{\prime}{a}bc(1+\varrho)} + \frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{\overset{\prime}{a}bc(1+\varrho)^2} + \frac{\overset{\prime\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{\overset{\prime}{a}bc(1+\varrho)^3}$ + &c is equal to $\overset{\prime}{A}BC$, will become equal to $\frac{\overset{\prime}{a}bc}{abc(1+\varrho)}(1+\overset{\prime}{A}BC) = \frac{1+\overset{\prime}{A}BC}{(1+\varrho)} \times \frac{\overset{\prime}{a}}{a}$. In like manner, it is evident that, by leaving out the common multiple $\frac{2s}{6}$, the fourth perpendicular series, or $\frac{a\overset{\prime}{b}\overset{\prime}{c}}{abc(1+\varrho)} + \frac{\overset{\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc(1+\varrho)^2} + \frac{\overset{\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc(1+\varrho)^3}$ + &c, is equal to $\frac{\underset{\prime}{a}}{a}\left[\frac{a\overset{\prime}{b}\overset{\prime}{c}}{\underset{\prime}{a}bc(1+\varrho)} + \frac{\overset{\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{\underset{\prime}{a}bc(1+\varrho)^2} + \frac{\overset{\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{\underset{\prime}{a}bc(1+\varrho)^3} + \&c\right]$: which, since $\frac{a\overset{\prime}{b}\overset{\prime}{c}}{\underset{\prime}{a}bc(1+\varrho)} + \frac{\overset{\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{\underset{\prime}{a}bc(1+\varrho)^2} + \frac{\overset{\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{\underset{\prime}{a}bc(1+\varrho)^3}$ + &c is equal to $\underset{\prime}{A}BC$, will become equal to

$\underset{\prime}{A}BC\times\frac{\underset{\prime}{a}}{a}$. Therefore the sum of these two series, taking the former of them as negative, agreeably to the general expression in the text, will be $-\frac{1+\overset{\prime}{A}BC}{(1+\varrho)}\times\frac{\overset{\prime}{a}}{a}+\underset{\prime}{A}BC\times\frac{\underset{\prime}{a}}{a}$; and which being multiplied by their common multiple $\frac{2s}{6}$, will produce $-\frac{s}{3a}\times\left[\frac{(1+\overset{\prime}{A}BC)\overset{\prime}{a}}{(1+\varrho)}-\underset{\prime}{A}BC.\underset{\prime}{a}\right]$ for the value of the same. The same method must be pursued in order to find the sum of the next two series, and also of the last two: but enough has been here said to enable the reader to perform these operations, without stating the process at large.

I would, however, observe that these several series may be expressed by *other* formulæ than those given in the text: for, the third perpendicular series, taking all the terms as affirmative and leaving out the common multiple $\frac{2s}{6}$, is also equal to $\frac{\underset{\prime}{b}\underset{\prime}{c}}{bc}\times\left[\frac{\overset{\prime}{a}bc}{a\underset{\prime}{b}\underset{\prime}{c}(1+\varrho)}+\frac{\overset{\prime\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{a\underset{\prime}{b}\underset{\prime}{c}(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{a\underset{\prime}{b}\underset{\prime}{c}(1+\varrho)^3}+\&c\right]$; which (since the sum of the terms within the brackets is equal to $A\underset{\prime}{B}\underset{\prime}{C}$) will become equal to $A\underset{\prime}{B}\underset{\prime}{C}\times\frac{\underset{\prime}{b}\underset{\prime}{c}}{bc}$: and the fourth perpendicular series, leaving out the common multiple $\frac{2s}{6}$, is also equal to $\frac{a\overset{\prime}{b}\overset{\prime}{c}}{abc(1+\varrho)}\times\left[1+\frac{\overset{\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{a\overset{\prime}{b}\overset{\prime}{c}(1+\varrho)}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{a\overset{\prime}{b}\overset{\prime}{c}(1+\varrho)^2}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime\prime}{b}\overset{\prime\prime\prime\prime}{c}}{a\overset{\prime}{b}\overset{\prime}{c}(1+\varrho)^3}+\&c\right]$: which

and sixth is equal to $\frac{s}{6b}\left[\frac{(1+AB'C)b'}{(1+\varrho)} - AB_{\prime}C.b_{\prime}\right]$; and the sum of the seventh and eighth is equal to

(since the sum of the terms within the brackets is equal to $1+AB'C'$) becomes equal to $\frac{1+AB'C'}{(1+\varrho)} \times \frac{b'c'}{bc}$. Consequently, the sum of the third and fourth perpendicular series given in the text will also be denoted by $+\frac{s}{3bc}\left[\frac{(1+AB'C')b'c'}{(1+\varrho)} - AB_{\prime}C_{\prime}.b_{\prime}c_{\prime}\right]$; and may be substituted at pleasure for the same. And universally we shall find that

$$\frac{1+A'BC}{(1+\varrho)} \cdot \frac{a'}{a} = AB_{\prime}C_{\prime} \cdot \frac{b_{\prime}c_{\prime}}{bc}$$

$$A_{\prime}BC \cdot \frac{a_{\prime}}{a} = \frac{1+AB'C'}{(1+\varrho)} \cdot \frac{b'c'}{bc}$$

$$\frac{1+AB'C}{(1+\varrho)} \cdot \frac{b'}{b} = A_{\prime}BC_{\prime} \cdot \frac{a_{\prime}c_{\prime}}{ac}$$

$$AB_{\prime}C \cdot \frac{b_{\prime}}{b} = \frac{1+A'BC'}{(1+\varrho)} \cdot \frac{a'c'}{ac}$$

$$\frac{1+ABC'}{(1+\varrho)} \cdot \frac{c'}{c} = A_{\prime}B_{\prime}C \cdot \frac{a_{\prime}b_{\prime}}{ab}$$

$$ABC_{\prime} \cdot \frac{c_{\prime}}{c} = \frac{1+A'B'C}{(1+\varrho)} \cdot \frac{a'b'}{ab}$$

It therefore appears that *each* series may be summed up in *two* different ways, and that we may adopt either mode of expression for the value of the same. For the sake of uniformity I have, in this and the following problems, kept to those which are given in the text: but Mr. Morgan (by merely changing these expressions, one for the other, according to the seniority of the lives, and then treating them as *different* quantities,) has thrown an air of obscurity and confusion throughout the whole of his investigations; as I shall point out in the *Observations* at the end of this problem. See § 251.

$\frac{s}{6c}\left[\frac{(1+AB\overset{\prime}{C})\overset{\prime}{c}}{(1+\varrho)}-AB\underset{\prime}{C}.\underset{\prime}{c}\right]$. Consequently, the total present value of the given sum to be received on the above contingency, will be equal to s multiplied into

$$\frac{1-\varrho ABC}{3(1+\varrho)}-\frac{1}{3a}\left[\frac{(1+\overset{\prime}{A}BC)\overset{\prime}{a}}{(1+\varrho)}-\underset{\prime}{A}BC.\underset{\prime}{a}\right]+\frac{1}{6b}\times\left[\frac{(1+A\overset{\prime}{B}C)\overset{\prime}{b}}{(1+\varrho)}-A\underset{\prime}{B}C.\underset{\prime}{b}\right]+\frac{1}{6c}\left[\frac{(1+AB\overset{\prime}{C})\overset{\prime}{c}}{(1+\varrho)}-AB\underset{\prime}{C}.\underset{\prime}{c}\right].$$

As this formula will often be referred to, in the subsequent problems, it will be convenient to denote it by a more simple expression; therefore let it be represented by $.\mathbf{A}^{BC}$: that is, let $.\mathbf{A}^{BC}$ denote the present value of £1 to be received on the above contingency: consequently $s\times.\mathbf{A}^{BC}$ will denote the present value of the given sum under the same circumstances.

COROLLARY I.

§ 248. If it were required to find the present value of the given sum payable on the decease of B, provided he be the first that fails of the three lives, we may readily obtain such value by substituting A for B, and B for A, in the investigation of the problem. Whence, the present value required would come out equal to s multiplied into $\frac{1-\varrho ABC}{3(1+\varrho)}+$

$$\frac{1}{6a}\left[\frac{(1+\overset{\prime}{A}BC)\overset{\prime}{a}}{(1+\varrho)}-\underset{\prime}{A}BC.\underset{\prime}{a}\right]-\frac{1}{3b}\left[\frac{(1+A\overset{\prime}{B}C)\overset{\prime}{b}}{(1+\varrho)}-\right.$$

$$A\underset{\prime}{B}C.\underset{\prime}{b}\Big] + \frac{1}{6c}\left[\frac{(1+AB\overset{\prime}{C})\overset{\prime}{c}}{(1+\varrho)} - AB\underset{\prime}{C}.\underset{\prime}{c}\right] = \mathfrak{B}^{AC}.^{*}$$

By a similar process it will be found that the present value of the given sum payable on the decease of C, provided he be the first that fails of the three lives, is equal to s multiplied into $\frac{1-\varrho ABC}{3(1+\varrho)} + \frac{1}{6a}\left[\frac{(1+\overset{\prime}{A}BC)\overset{\prime}{a}}{(1+\varrho)} - \underset{\prime}{A}BC.\underset{\prime}{a}\right] + \frac{1}{6b}\left[\frac{(1+A\overset{\prime}{B}C)\overset{\prime}{b}}{(1+\varrho)} - A\underset{\prime}{B}C.\underset{\prime}{b}\right] - \frac{1}{3c} \times \left[\frac{(1+AB\overset{\prime}{C})\overset{\prime}{c}}{(1+\varrho)} - AB\underset{\prime}{C}.\underset{\prime}{c}\right] = \mathfrak{C}^{AB}.^{*}$ As these formulæ may occasionally be found of use, I have thought proper to insert them here.

COROLLARY II.

§ 240. If the three lives are equal, or of the same

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[*] I have represented these complex formulæ by the more simple quantities $\mathfrak{B}^{AC}$ and $\mathfrak{C}^{AB}$, for the sake of a more convenient reference; and the following process will show with how little trouble they may be converted into each other.——Let us make $\frac{1}{a}\left[\frac{(1+\overset{\prime}{A}BC)\overset{\prime}{a}}{(1+\varrho)} - \underset{\prime}{A}BC.\underset{\prime}{a}\right] = \mathrm{a}$, $\frac{1}{b}\left[\frac{(1+A\overset{\prime}{B}C)\overset{\prime}{b}}{(1+\varrho)} - A\underset{\prime}{B}C.\underset{\prime}{b}\right] = \mathrm{b}$, and $\frac{1}{c}\left[\frac{(1+AB\overset{\prime}{C})\overset{\prime}{c}}{(1+\varrho)} - AB\underset{\prime}{C}.\underset{\prime}{c}\right] = \mathrm{c}$; then will

$$\mathfrak{A}^{BC} = \frac{1-\varrho ABC}{3(1+\varrho)} - \frac{\mathrm{a}}{3} + \frac{\mathrm{b}}{6} + \frac{\mathrm{c}}{6}$$

$$\mathfrak{B}^{AC} = \frac{1-\varrho ABC}{3(1+\varrho)} + \frac{\mathrm{a}}{6} - \frac{\mathrm{b}}{3} + \frac{\mathrm{c}}{6}$$

$$\mathfrak{C}^{AB} = \frac{1-\varrho ABC}{3(1+\varrho)} + \frac{\mathrm{a}}{6} + \frac{\mathrm{b}}{6} - \frac{\mathrm{c}}{3}.$$

age A, the last three terms in each of the above expressions destroy each other; and the formula is then reduced to $\frac{s}{3} \times \frac{1-\varrho A.{}'A}{(1+\varrho)}$: an expression which denotes one third of the present value of the given sum to be received on the extinction of the three joint lives.

COROLLARY III.*

§ 250. If the contingency, on which the sum is to be received, continues only for a given *term* ($=n$), the present value of such sum will be equal to the sum of the first n terms of the several series given in the problem: the method of determining which will be manifest from the many examples which have preceded.†

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* Morgan, Prob. 31.

† We may in general obtain a near value of the sum, in this case, by the help of M. De Moivre's hypothesis, which will save much time and trouble. For, the series given in the problem is evidently equal to

$$\frac{s}{\delta(1+\varrho)} \times \frac{a-a'}{a}\left[\frac{2b'c}{bc}+\frac{2b'c'}{bc}+\frac{b'c}{bc}+\frac{bc'}{bc}\right]+$$
$$\frac{s}{\delta(1+\varrho)^2} \times \frac{a'-a''}{a}\left[\frac{2b'c'}{bc}+\frac{2b''c''}{bc}+\frac{b''c'}{bc}+\frac{b'c''}{bc}\right]+$$
$$\frac{s}{\delta(1+\varrho)^3} \times \frac{a''-a'''}{a}\left[\frac{2b''c''}{bc}+\frac{2b'''c'''}{bc}+\frac{b'''c''}{bc}+\frac{b''c'''}{bc}\right]+$$
$$\&c \qquad\qquad \&c \qquad\qquad \&c,$$

which being continued to n terms will be the required value of the given sum. But, since the quantities $a-a'$, $a'-a''$, $a''-a'''$, &c are supposed to be equal and uniform (that is, equal to $\delta = \frac{a-\alpha}{n}$) the sum of the first n terms of these several perpendicular series will

Observations on Mr. Morgan's Method of investigating this Problem.

§ 251. The motives, which induced me to notice (in page 195) the strange method which Mr. Morgan has adopted in summing up the several series arising from the investigation of the twenty-seventh problem, must be my apology here for again detaining the reader whilst I expose the equally diffuse and obscure manner which he has also adopted in summing up the several series arising from the investigation of this problem.

It will be seen, from an inspection of the series in page 207, that such series may be expressed in the following manner:

$$\frac{s}{(1+\varrho)}\left[\frac{(a-a')bc}{3abc}+\frac{(a-a')b'c}{6abc}+\frac{(a-a')bc'}{6abc}+\frac{(a-a')b'c'}{3abc}\right]+$$

$$\frac{s}{(1+\varrho)^2}\left[\frac{(a'-a'')b'c'}{3abc}+\frac{(a'-a'')b''c'}{6abc}+\frac{(a'-a'')b'c''}{6abc}+\frac{(a'-a'')b''c''}{3abc}\right]+$$

$$\frac{s}{(1+\varrho)^3}\left[\frac{(a''-a''')b''c''}{3abc}+\frac{(a''-a''')b'''c''}{6abc}+\frac{(a''-a''')b''c'''}{6abc}+\frac{(a''-a''')b'''c'''}{3abc}\right]+$$

&c &c &c;

and this is the way in which Mr. Morgan has thought

be equal to $s.\frac{a-\alpha}{an}$ multiplied into $\frac{1+(BC)^{t-1}}{3(1+\varrho)}+\frac{(BC)^{t}}{3}+\frac{1+(B'C)^{t-1}}{6(1+\varrho)}\cdot\frac{b'}{b}+\frac{(B_{\prime}C)^{t}}{6}\cdot\frac{b}{b'}$. If A is the youngest of the three lives, and differs much from the ages of both the other lives, this formula will be tolerably correct, even if the several perpendicular series are continued to the utmost extent of human life; and will consequently serve in such cases to find a *near* value of the sum for the whole continuance of the lives: but it must always be used with proper caution.

proper to represent the present value of the given sum, to be received on the contingency mentioned in the problem.*

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* In this investigation I have adopted the same symbols which have been used throughout the whole of this work: but, lest I may be accused of having misrepresented Mr. Morgan, I shall here give his own solution, and in his own style.

Let r denote £1 increased by its interest for a year. Let a denote the number of persons living at the age of A; and $\overset{\prime}{a}, \overset{\prime\prime}{a}, \overset{\prime\prime\prime}{a}$, &c, the decrements of life at the end of the first, second, third, &c, years from the age of A. Let b denote the number of persons living at the age of B; and m, n, o, p, &c, the number living at the end of one, two, three, &c, years from that age. Let c denote the number of persons living at the age of C; and d, e, f, &c, the number living at the end of one, two, three, &c, years from that age. Let F denote the value of an annuity on a life one year younger than B; and β the number of persons living at that age. Let K denote the value of an annuity on a life one year younger than C; and $\varkappa$ the number of persons living at that age. Let P denote the value of an annuity one year older than B: and T, the value of an annuity one year older than C. And let the given sum be denoted by S.

Then (in the words of Mr. Morgan, *Phil. Trans.* for 1791, p. 249) the whole value of the given sum will be $= \frac{S}{3abc} \times \left[\frac{\overset{\prime}{a}bc}{r} + \frac{\overset{\prime\prime}{a}md}{r^2} + \frac{\overset{\prime\prime\prime}{a}en}{r^3} + \&c\right] + \frac{S}{6abc} \times \left[\frac{\overset{\prime}{a}mc}{r} + \frac{\overset{\prime\prime}{a}nd}{r^2} + \frac{\overset{\prime\prime\prime}{a}oe}{r^3} + \&c\right] + \frac{S}{6abc} \times \left[\frac{\overset{\prime}{a}db}{r} + \frac{\overset{\prime\prime}{a}em}{r^2} + \frac{\overset{\prime\prime}{a}fn}{r^3} + \&c\right] + \frac{S}{3abc} \times \left[\frac{\overset{\prime}{a}md}{r} + \frac{\overset{\prime\prime}{a}en}{r^2} + \frac{\overset{\prime\prime\prime}{a}fo}{r^3} + \&c\right]$. But, the first of these series (he observes) is $= \frac{S}{3} \times \frac{\beta\varkappa(\text{FK} - \text{AFK})}{bc} - \frac{S}{3r} \times (\text{BC} - \text{ABC})$; the second is $= \frac{S}{6} \times \frac{\varkappa(\text{EK} - \text{AEK})}{c} - \frac{S}{6r} \times$

§ 252. In order to sum up these four collateral series he proceeds in the following manner. The

$\frac{m(\text{PC}-\text{APC})}{b}$; the third is $= \frac{\text{S}}{6} \times \frac{\beta(\text{FC}-\text{AFC})}{b} - \frac{\text{S}}{6r} \times \frac{d(\text{BT}-\text{ABT})}{c}$; the fourth is $= \frac{\text{S}}{3} \times (\text{BC}-\text{ABC}) - \frac{\text{S}}{3r} \times \frac{md(\text{PT}-\text{APT})}{bc}$. Therefore these several expressions being added together will be found $=$ S into $\frac{\varkappa}{3c} \times \left[\frac{\beta(\text{FK}-\text{AFK})}{b} + \frac{\text{BK}-\text{ABK}}{2}\right] + \frac{\beta}{6b} \times (\text{FC}-\text{AFC}) + \frac{r-1}{3r} \times (\text{BC}-\text{ABC}) - \frac{m(\text{PC}-\text{APC})}{6br} - \frac{d}{3cr} \times \left[\frac{\text{BT}-\text{ABT}}{2} + \frac{m(\text{PT}-\text{APT})}{b}\right]$.

Thus far Mr. Morgan.——And these several values, being expressed by the characters made use of in the present work, will be found to agree with the process given in the text. But it may be demonstrated that

$$\frac{\beta\varkappa.\text{FK}}{bc} - \frac{\text{BC}}{r} = \frac{1}{r}$$

$$\frac{\varkappa.\text{BK}}{c} - \frac{m.\text{PC}}{br} = \frac{m}{br}$$

$$\frac{\beta.\text{FC}}{b} - \frac{d.\text{BT}}{cr} = \frac{d}{cr}$$

$$\text{BC} - \frac{dm.\text{PT}}{bcr} = \frac{dm}{bcr}:$$

whence, the complex formula above given may be divested of all those expressions involving *two* joint lives, and may be more simply expressed by S multiplied into $\frac{1-(r-1)\text{AFC}}{3r} + \frac{1}{3bc} \times \left[\frac{(1+\text{APT})dm}{r} - \text{AFK}.\beta\varkappa\right] + \frac{1}{6b}\left[\frac{(1+\text{APC})m}{r} - \text{AFC}\,\beta\right] + \frac{1}{6c}\left[\frac{(1+\text{ABT})d}{r} - \text{ABK}.\varkappa\right]$: which corresponds with the formula given in the text.

first of them he expands (as in *Phil. Trans.* for 1789, page 44) into the more complex series

$$\frac{s}{(1+\varrho)}\left[\frac{bc}{3bc}-\frac{a'bc}{3abc}\;\ldots\ldots\ldots\right]+$$
$$\frac{s}{(1+\varrho)^2}\left[\frac{b'c'}{3bc}-\frac{a''b'c'}{3abc}-\frac{b'c'}{3bc}+\frac{a'b'c'}{3abc}\right]+$$
$$\frac{s}{(1+\varrho)^3}\left[\frac{b''c''}{3bc}-\frac{a'''b''c''}{3abc}-\frac{b''c''}{3bc}+\frac{a''b''c''}{3abc}\right]+$$
$$\&c \qquad \&c \qquad \&c,$$

the sum of all which (independent of the common multiple $\frac{s}{3}$) he makes equal to $B_{\prime}C_{\prime} \times \frac{b_{\prime}c_{\prime}}{bc} - AB_{\prime}C_{\prime} \times \frac{b_{\prime}c_{\prime}}{bc} - \frac{BC}{(1+\varrho)} + \frac{ABC}{(1+\varrho)}$.

By proceeding in a similar manner, he makes the *second* collateral series, in page 214, (independent of the common multiple $\frac{s}{3}$) equal to $BC_{\prime} \times \frac{c_{\prime}}{c} - ABC_{\prime} \times \frac{c_{\prime}}{c} - \frac{B'C}{(1+\varrho)} \times \frac{b'}{b} + \frac{AB'C}{(1+\varrho)} \times \frac{b'}{b}$: the *third* equal to $B_{\prime}C \times \frac{b_{\prime}}{b} - AB_{\prime}C \times \frac{b_{\prime}}{b} - \frac{BC'}{(1+\varrho)} \times \frac{c'}{c} + \frac{ABC'}{(1+\varrho)} \times \frac{c'}{c}$: and the *fourth* equal to $BC - ABC - \frac{B'C'}{(1+\varrho)} \times \frac{b'c'}{bc} + \frac{AB'C'}{(1+\varrho)} \times \frac{b'c'}{bc}$. Consequently, the sum of the four collateral series in page 214 (or the total present value of the given sum to be received on the contingency mentioned in the problem) will, as he observes, be found equal to s multiplied into $B_{\prime}C_{\prime} \times \frac{b_{\prime}c_{\prime}}{3bc}$ —

$$A\underset{,}{B}\underset{,}{C} \times \frac{\underset{,}{b}\underset{,}{c}}{3bc} - \frac{BC}{3(1+\varrho)} + \frac{ABC}{3(1+\varrho)} + B\underset{,}{C} \times \frac{\underset{,}{c}}{6c} -$$

$$AB\underset{,}{C} \times \frac{\underset{,}{c}}{6c} - \frac{\overset{,}{B}C}{(1+\varrho)} \times \frac{\overset{,}{b}}{6b} + \frac{A\overset{,}{B}C}{(1+\varrho)} \times \frac{\overset{,}{b}}{6b} + \underset{,}{B}C \times$$

$$\frac{\underset{,}{b}}{6b} - A\underset{,}{B}C \times \frac{\underset{,}{b}}{6b} - \frac{B\overset{,}{C}}{(1+\varrho)} \times \frac{\overset{,}{c}}{6c} + \frac{AB\overset{,}{C}}{(1+\varrho)} \times \frac{\overset{,}{c}}{6c} +$$

$$\frac{BC}{3} - \frac{ABC}{3} - \frac{\overset{,}{B}\overset{,}{C}}{(1+\varrho)} \times \frac{\overset{,}{b}\overset{,}{c}}{3bc} + \frac{A\overset{,}{B}\overset{,}{C}}{(1+\varrho)} \times \frac{\overset{,}{b}\overset{,}{c}}{3bc}.$$

§ 253. Now, a moment's attention to the steps of this process will convince us that all those quantities, in which *two* joint lives are involved, are *unnecessarily* introduced into the above formula; since they evidently destroy each other, and may consequently be made to vanish altogether. For,

$$\underset{,}{B}\underset{,}{C} \times \frac{\underset{,}{b}\underset{,}{c}}{bc} - \frac{BC}{(1+\varrho)} = \frac{1}{(1+\varrho)}\text{[*]}$$

$$B\underset{,}{C} \times \frac{\underset{,}{c}}{c} - \frac{\overset{,}{B}C}{(1+\varrho)} \times \frac{\overset{,}{b}}{b} = \frac{\overset{,}{b}}{b(1+\varrho)}$$

$$\underset{,}{B}C \times \frac{\underset{,}{b}}{b} - \frac{B\overset{,}{C}}{(1+\varrho)} \times \frac{\overset{,}{c}}{c} = \frac{\overset{,}{c}}{c(1+\varrho)}$$

$$BC - \frac{\overset{,}{B}\overset{,}{C}}{(1+\varrho)} \times \frac{\overset{,}{b}\overset{,}{c}}{bc} = \frac{\overset{,}{b}\overset{,}{c}}{bc(1+\varrho)}:$$

whence, the complex formula given by Mr. Morgan may be reduced to s multiplied into $\frac{1-\varrho ABC}{3(1+\varrho)}$ +

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[*] It is evident, *on inspection*, that this is the value of the first and third collateral series in page 217; because all the subsequent terms *after the first term* manifestly destroy each other. Consequently the sum of those two collateral series will be equal to the

$$\frac{1}{3bc}\left[\frac{(1+A\overset{\prime\prime}{B}\overset{\prime}{C})\overset{\prime\prime}{l}\overset{\prime}{c}}{(1+\varrho)} - A\underset{\prime}{B}\underset{\prime}{C}.\underset{\prime\prime}{b}c\right] + \frac{1}{6b}\left[\frac{(1+A\overset{\prime}{B}C)\overset{\prime}{b}}{(1+\varrho)} - \right.$$

$$\left. A\underset{\prime}{B}C.\underset{\prime}{b}\right] + \frac{1}{6c}\left[\frac{(1+AB\overset{\prime}{C})\overset{\prime}{c}}{(1+\varrho)} - AB\underset{\prime}{C}\,\underset{\prime}{c}\right]$$: and which is the very same as that which I have given in page

first term only: that is, (independent of the common multiple $\frac{s}{3}$) equal to $\frac{1}{(1+\varrho)} \times \frac{bc}{bc} = \frac{1}{(1+\varrho)}$. In like manner may be demonstrated the truth of the remaining equations.

I shall however make another observation here, which I think of considerable importance. Although these equations are strictly and mathematically correct, yet (owing to the imperfection of the Tables which show the value of Life Annuities) they oftentimes vary in a material degree when solved *arithmetically*: an inaccuracy which, being repeated so frequently in the *same* formula, must render the solution, according to Mr. Morgan's investigation, extremely erroneous.

Thus, the compound quantity $B\underset{\prime}{C}.\frac{\underset{\prime}{b}c}{bc} - \frac{BC}{(1+\varrho)}$ is equal to $\frac{1}{(1+\varrho)}$, and (if simplicity and elegance were out of the question, and no regard were paid to the labour and intricacy of the solution) might be substituted instead. Let us, however, reduce these expressions to *numbers*: for which purpose let us take the age of B 15 years, the age of C 18 years, the rate of interest 4 per cent, and the probabilities of living as at Northampton. In this case we should have $\frac{1}{(1+\varrho)} = {\cdot}962$; but the value of $B\underset{\prime}{C}.\frac{\underset{\prime}{b}c}{bc} - \frac{BC}{(1+\varrho)}$ is equal to $13{\cdot}566 - 12{\cdot}614 = {\cdot}952$ only: a difference which arises from the want of more enlarged tables of the value of annuities on two joint lives, and which will have a considerable effect upon the proper solution of the problem. This remark extends to *all* the formulæ inserted by Mr. Morgan in the several volumes of the *Philosophical Transactions*; and will show that those formulæ, however correct in principle, are of little or no use in *practice*.

211, except as to the substitution of the second term $+\frac{1}{3bc}\left[\frac{(1+ABC)b'c'}{(1+\rho)} - AB_{,}C_{,}.b_{,,}c_{,,}\right]$ for its equal $-\frac{1}{3a}\left[\frac{(1+A'BC)a'}{(1+\rho)} - A_{,}BC\,a_{,}\right]$, agreeably to what I have said in the note in page 210.

From this it will be readily seen that Mr. Morgan's formula consists of *double the number of terms* that are requisite for solving the problem; whereby it is rendered not only unnecessarily intricate and laborious, but also liable to considerable error in the *arithmetical* solution. Indeed it is manifest, from the nature of the series in page 207, that no expression involving *two* joint lives can properly arise: and it is singular that this consideration did not induce Mr. Morgan (if his object were perspicuity and information) to retrace the steps of his investigation, in order to render the process at once more simple and clear.

§ 254. But, there is another part of Mr. Morgan's investigation on which I think it equally necessary to make some observations. He states that the theorem which he has deduced, as above, "gives the exact "value when either B or C is the oldest of the three "lives; but when A is the oldest, it will be *necessary**

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* It certainly is not *necessary* to change the symbols; since, in whatever manner the characters are varied, we must come to the same general series at last: neither does such a measure facilitate the means of investigation. Indeed I am at a loss to conceive why Mr. Morgan should have attempted to render so obscure this interesting branch of the subject. As a display of ingenuity it must completely fail: since, his papers will be read with very different emotions from those of admiration.

‘ to change the symbols &c:” and he has deduced mother formula, for the solution of the problem in his case, involving *other* characters and symbols which render the investigation still more confused. From all which he would seem to imply that the first formula was not applicable to the second case.

But the second formula which he has deduced is still the *same* (mixed however with the useless and extraneous quantities above alluded to) as that which I have given in page 211; except as to the exchange of the last term $+\frac{1}{6c}\left[\frac{(1+AB\acute{C})\acute{c}}{(1+\varrho)}-ABC_{\prime}.c_{\prime}\right]$ for its equal $-\frac{1}{6ab}\left[\frac{(1+\acute{A}\acute{B}C)\acute{a}\acute{b}}{(1+\varrho)}-A_{\prime}B_{\prime}C.a_{\prime}b_{\prime}\right]$, agreeably to what I have said in the note in page 210.* And it may be here useful to remark that the formula, which I have given in page 211, is *universally* true, and does not depend on the seniority of the lives concerned. The substitutions above mentioned are mere arbitrary arrangements for the numerical solution of the problem; which may be occasionally used, or not, at the option of the calculator.

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* Mr. Morgan's formula, expressed by his own ambiguous characters, is in this case equal to S multiplied into $\frac{\beta}{3b}\times\left[\frac{\alpha(\mathrm{HF}+\frac{1}{2}\mathrm{HFC})}{a}-(\mathrm{AF}+\frac{1}{2}\mathrm{AFC})\right]+\frac{1}{6}\times\left[\frac{\alpha(\mathrm{HB}+2\mathrm{HBC})}{a}-(\mathrm{AB}+2\mathrm{ABC})\right]+\frac{1}{3r}\times\left[\frac{s(\mathrm{BN}-\mathrm{BNC})}{a}-(\mathrm{AB}-\mathrm{ABC})\right]+\frac{m}{6br}\times\left[\frac{s(\mathrm{PN}-\mathrm{PNC})}{a}-(\mathrm{AP}-\mathrm{APC})\right]$. Where H and N denote the value of an annuity on a life one year younger and one year older than A respectively; and where α and s denote the number of persons living at those ages respectively: the other symbols being the same as in the note in page 215.

But this complex formula, being divested of those *useless* quantities which express the values of annuities on *two* joint lives, may be reduced to the simple terms alluded to in the text. See the original formula in *Phil. Trans.* for 1791, page 251: or in Price's *Obs. on Rev. Pay.* Problem I in note (P).

§ 255. It is easy to perceive, from what is here said, that the formula (denoting the value of the sum depending on the contingency in this problem) may be varied in *three* different ways, merely by the manner of summing up the several collateral series given in page 207; agreeably to the method laid down in the note in page 209. But such variation is by no means necessary or useful. Mr. Morgan has, in a very circuitous and diffuse manner, demonstrated only *two* of these ways:* but, by this additional attempt at elucidation, he has certainly rendered the subject still more confused. The introduction, indeed, of *unnecessary quantities* into any investigation (but more particularly the retaining of them in any resulting formulæ) and the capricious changing of the symbols employed, ought to be universally reprobated; not only as subversive of the true ends of science (whose object is information and not mystery), but also as destructive of all good taste in mathematical reasoning.

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* That is, when B or C is the oldest of the three lives, and when A is the oldest of the three lives. But, in none of the problems (inserted by him in the *Phil. Trans.* for 1788, 1789, or 1791) do the values depend on the *seniority* of the lives concerned: for, *either* of the formulæ deduced by him, in the respective problems, will be *equally* correct whether A, B, or C be the oldest life.

I have thought it proper to make these observations in this place, because the present problem is of considerable importance in enabling us to determine the value of many of the subsequent problems; and is made use of by him for that purpose: therefore the remarks here made will equally apply to those problems in which Mr. Morgan has so used it. Indeed, I believe *there is not a single problem* inserted by him in *any* of his papers in the *Philosophical Transactions*, respecting the value of Contingent Assurances, wherein this prolix and confused method has not been adopted, in order to determine the same.

PROBLEM XXX.*

§ 256. To determine the present value of a given sum payable on the decease of A, provided he be the *second* that fails of three given lives A, B, C.

SOLUTION.

The sum may be received at the end of the first year, on the happening of either of three different events: 1° that all the three lives fail in that year, A having died second: 2° that A and B fail in the year,

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* Morgan, Prob. 20; and in Phil. Trans. for 1791, Prob. 2, page 253.

A having died last, and that C lives to the end of it: 3° that A and C fail in the year, A having died last, and that B lives to the end of it. The probabilities of the happening of these several events are respectively $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{3abc}$, $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b})\overset{\prime}{c}}{2abc}$ and $\frac{(a-\overset{\prime}{a}).(c-\overset{\prime}{c})\overset{\prime}{b}}{2abc}$: which, being added together and multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, the given sum may be received on the happening of either of seven different events: 1° that all the three lives become extinct in the year, A having died second: 2° that A and B both fail in the year, A having died last, and that C lives to the end of it: 3° that A and C both fail in the year, A having died last, and that B lives to the end of it: 4° that A and B both fail in the year, B having died last, and C having failed in either of the preceding years: 5° that A and C both fail in the year, C having died last, and B having failed in either of the preceding years: 6° that only A dies in the year, B living to the end of it, and C having died in either of the preceding years: 7° that only A dies in the year, C living to the end of it, and B having died in either of the preceding years. The probabilities of the happening of these several events in the second year, are respectively $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{3abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{c}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{b}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{2ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$,

$\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{c}}{ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$: which, being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner it will be found that the probabilities of the happening of these several events in the third year will be respectively denoted by $\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a}).(\overset{\prime\prime}{b}-\overset{\prime\prime\prime}{b}).(\overset{\prime\prime}{c}-\overset{\prime\prime\prime}{c})}{3abc}$, $\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a}).(\overset{\prime\prime}{b}-\overset{\prime\prime\prime}{b})\overset{\prime\prime\prime}{c}}{2abc}$, $\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a}).(\overset{\prime\prime}{c}-\overset{\prime\prime\prime}{c})\overset{\prime\prime\prime}{b}}{2abc}$, $\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a}).(\overset{\prime\prime}{b}-\overset{\prime\prime\prime}{b})}{2ab}\left(1-\frac{\overset{\prime\prime}{c}}{c}\right)$, $\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a}).(\overset{\prime\prime}{c}-\overset{\prime\prime\prime}{c})}{2ac}\left(1-\frac{\overset{\prime\prime}{b}}{b}\right)$, $\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a})\overset{\prime\prime\prime}{b}}{ab}\left(1-\frac{\overset{\prime\prime}{c}}{c}\right)$, and $\frac{(\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a})\overset{\prime\prime\prime}{c}}{ac}\times\left(1-\frac{\overset{\prime\prime}{b}}{b}\right)$: which, being added together and multiplied by $s(1+\varrho)^{-3}$, will give the expectation of receiving the sum at the end of the third year: and so on for every subsequent year to the utmost extent of human life: the sum of all which yearly expectations will be the total present value of the given sum to be received on the above contingency.

§ 257. Now if these several annual expectations be reduced to their least terms, and then arranged under each other as in the preceding problem, they will be found to form sixteen collateral series; the sum of all which will be equal to s multiplied into

$$\frac{1-\varrho AB}{2(1+\varrho)}-\frac{1+\overset{\prime}{A}B}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a}+\underset{\prime}{A}B.\frac{\underset{\prime}{a}}{2a}+\frac{1-\varrho AC}{2(1+\varrho)}-\frac{1+\overset{\prime}{A}C}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a}$$

$$+\underset{\prime}{A}C.\frac{\underset{\prime}{a}}{2a}-\frac{2}{3}\times\frac{1-\varrho ABC}{(1+\varrho)}+\frac{1+\overset{\prime}{A}BC}{3(1+\varrho)}.\frac{2\overset{\prime}{a}}{a}-\underset{\prime}{A}BC.\frac{\underset{\prime}{2a}}{3a}$$
$$-\frac{1+A\overset{\prime}{B}C}{3(1+\varrho)}.\frac{\overset{\prime}{b}}{b}+\underset{\prime}{A}BC.\frac{\underset{\prime}{b}}{3b}-\frac{1+AB\overset{\prime}{C}}{3(1+\varrho)}.\frac{\overset{\prime}{c}}{c}+AB\underset{\prime}{C}.\frac{\underset{\prime}{c}}{3c}.$$

But, the first three terms here given are, by Prob. XXVII, equal to $.\mathcal{A}^{B}$; the next three terms are, by the second corollary to the same problem, equal to $.\mathcal{A}^{C}$; and the remaining seven terms are, by Prob. XXIX, equal to $-2.\mathcal{A}^{BC}$. Consequently, the total present value of the given sum depending on this contingency will be equal to $s\left[.\mathcal{A}^{B}+.\mathcal{A}^{C}-2.\mathcal{A}^{BC}\right]$.*

COROLLARY.

§ 258. If the three lives are equal, or of the same age A, then $.\mathcal{A}^{B}$ and $.\mathcal{A}^{C}$ will each of them become (as in Prob. XXVII, cor. 5) equal to $\frac{1-\varrho AA}{2(1+\varrho)}$; and $.\mathcal{A}^{BC}$ will become (as in Prob. XXIX, cor. 2) equal to $\frac{1-\varrho AAA}{3(1+\varrho)}$. Consequently the present value of the given sum will in such case be represented by $s \times \frac{1-\varrho(3AA-2AAA)}{3(1+\varrho)}$: or, by one third of the present value of the given sum payable on the extinction of any two out of the three given lives.

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* It will be evident from an inspection of the several quantities, whence this formula is deduced, that a more convenient one might have been formed for the *numerical* solution of the problem: similar to the method pursued in the 39th and subsequent problems. But the present simple formula, at the same time that it is more easily retained in the memory, has also the additional advantage of enabling us readily to discover any error which may arise in the process. I have therefore on this account inserted it: but the reader may, from the values of the several series above given, arrange the formula in such other manner as he may find most convenient to himself. The same observations will apply to many of the subsequent problems in this chapter.

PROBLEM XXXI.*

§ 259. To determine the present value of a given sum payable on the decease of A, provided he be the *last* which fails of three given lives A, B, C.

SOLUTION.

It is evident that the sum can be received at the end of the first year, only on the extinction of all the lives, A having died last, the probability of which is $\frac{(a-\overset{\prime}{a}) \times (b-\overset{\prime}{b}) \times (c-\overset{\prime}{c})}{3abc}$; and which, being multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years the sum may be received on the happening of either of four different events: 1° that all the lives fail in the year, A having died last: 2° that A and B fail in the year, A having died last, and that C fails in either of the preceding years: 3° that A and C fail in the year, A having died last, and that B fails in either of the preceding years: 4° that only A dies in the year, B and C having failed in either of the preceding years. The probabilities of these several events happening in the second year are respectively $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{3abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{2ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$, and $\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{a}\left(1-\frac{\overset{\prime}{b}}{b}\right)\times\left(1-\frac{\overset{\prime}{c}}{c}\right)$: which being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year.

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* Morgan, Prob. 21; and in Phil. Trans. for 1791, Prob. 3, page 256. Simpson, Prob. 30.

In like manner may be found the expectation of receiving the sum at the end of the third and every subsequent year to the utmost extent of human life: and, if these several yearly expectations be reduced to their lowest terms and arranged under each other, they will form eighteen collateral series, the sum of all which will be the present value required: and which will be found to be equal to s multiplied into $\frac{1-\varrho A}{(1+\varrho)} - \frac{1-\varrho AB}{2(1+\varrho)} + \frac{1+\overset{\prime}{A}B}{2(1+\varrho)}.\frac{\overset{\prime}{a}}{a} - \underset{\prime}{A}B.\frac{\underset{\prime}{a}}{2a} - \frac{1-\varrho AC}{2(1+\varrho)} + \frac{1+\overset{\prime}{A}C}{2(1+\varrho)}.\frac{\overset{\prime}{a}}{a} - \underset{\prime}{A}C.\frac{\underset{\prime}{a}}{2a} + \frac{1-\varrho ABC}{3(1+\varrho)} - \frac{1+\overset{\prime}{A}BC}{3(1+\varrho)}.\frac{\overset{\prime}{a}}{a} + \underset{\prime}{A}BC.\frac{\underset{\prime}{a}}{3a} + \frac{1+A\overset{\prime}{B}C}{6(1+\varrho)}.\frac{\overset{\prime}{b}}{b} - A\underset{\prime}{B}C.\frac{\underset{\prime}{b}}{6b} + \frac{1+AB\overset{\prime}{C}}{6(1+\varrho)}.\frac{\overset{\prime}{c}}{c} - AB\underset{\prime}{C}.\frac{\underset{\prime}{c}}{6c}$. But this expression is equal to $s\left[\frac{1-\varrho A}{(1+\varrho)} - \mathcal{A}^{B} - \mathcal{A}^{C} + \mathcal{A}^{BC}\right]$: which therefore denotes the present value required.

COROLLARY.

§ 260. If the three lives are equal, or of the same age A, the present value of the given sum will, agreeably to what has been said in the corollary to the last problem, be denoted by $s \times \frac{1-\varrho(3A-3AA+AAA)}{3(1+\varrho)}$: or, by one third of the present value of the given sum to be received on the extinction of the longest of the three lives.

SCHOLIUM.

§ 261. If the present values of the given sum, as found by Prob. XXIX, XXX, and XXXI, be added together, they will be found equal to the present value of the same sum to be received on the decease of A; that is, the sum of those three values will be equal to $s \times \frac{1-\varrho A}{(1+\varrho)}$: which proves the accuracy of the investigations.

PROBLEM XXXII.*

§ 262. To determine the present value of a given sum payable on the decease of A, provided he be the *first or second* that fails of three given lives A, B, C.

SOLUTION.

The given sum may be received at the end of the first year, on the happening of either of four different events: 1° that all the three lives fail in the year, A having died first or second: 2° that A and B both die in the year, and that C lives to the end of it: 3° that A and C both die in the year and that B lives to the end of it: 4° that only A dies in the year, and that B and C both live to the end of it. The probabilities of the happening of these several events are respectively denoted by $\frac{2(a-\overset{\prime}{a})\,(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{3abc}$, $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b})\overset{\prime}{c}}{abc}$, $\frac{(a-\overset{\prime}{a}).(c-\overset{\prime}{c})\overset{\prime}{b}}{abc}$, and $\frac{(a-\overset{\prime}{a})\overset{\prime}{b}\overset{\prime}{c}}{abc}$: and which being multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

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* Morgan, Prob. 24; and in Phil. Trans. for 1791, Prob. 5, page 261. Simpson's *Sup.* Prob. 40.

But, in the second and following years, the given sum may be received on the happening of either of eight different events: 1° that all the three lives fail in the year, A having died first or second: 2° that A and B both die in the year, and that C lives to the end of it: 3° that A and C both die in the year and that B lives to the end of it: 4° that only A dies in the year and that B and C both live to the end of it: 5° that A dies before B in the year, C having failed in either of the preceding years: 6° that A dies before C in the year, B having failed in either of the preceding years: 7° that only A dies in the year, C having died in either of the preceding years, and B living to the end of it: 8° that only A dies in the year, B having died in either of the preceding years, and C living to the end of it. The probabilities of the happening of these several events in the second year are denoted respectively by $\frac{2(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{3abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{c}}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{b}}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{2ab} \times \left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{c}}{ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$: which being added together, and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every succeeding year to the utmost extent of human life: the sum of all which expectations will be the total present value of the given sum, to be received on the above contingency.

These several yearly expectations, being reduced to their least terms and arranged under each other, will form sixteen collateral series;* the sum of all which is the present value required: and will be found equal to $s\left[\mathcal{A}^{B}+\mathcal{A}^{C}-\mathcal{A}^{BC}\right]$.†

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* It would be needless for me to swell the present work by inserting the *several* values of these series, since (from the examples which have preceded) they may be readily discovered by an inspection of the resulting formula. Few persons, perhaps, who read this part of the treatise, will be satisfied without going through the several steps of the process: in which case, such a measure would not be requisite. And I cannot here avoid the opportunity of recommending the reader to adopt this plan through all the subsequent problems; as he will find it an excellent *praxis* for this species of analysis.

† It may be here useful to remark that this formula may be ob-

COROLLARY.

§ 263. When the three lives are equal, or all of the same age A, this expression (agreeably to what has been said in the corollary to Prob. XXX) will become equal to $s \times \frac{2 - \varrho(3AA - AAA)}{3(1+\varrho)}$: or, equal to the difference between the value of an assurance of the given sum on the two joint lives, and one third of the value of an assurance of the same sum on the three joint lives.

PROBLEM XXXIII.*

§ 264. To determine the present value of a given sum payable on the decease of A, provided he be the *second or third* that fails of three given lives A, B, C,

SOLUTION.

The given sum may be received at the end of the

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tained by taking the *sum* of the values deduced from Prob. XXIX and XXX; or, by taking the *difference* between $s \times \frac{1 - \varrho A}{(1+\varrho)}$ and the value deduced from Prob. XXXI: which is evident from the nature of the questions. For, the sum is to be received on the death of A provided he dies *first or second:* or (which is the same thing) it is to be received on the death of A provided he does *not* die *third.*

* Morgan, Prob. 25; and in Phil. Trans. for 1791, Prob. 6, page 263.

first year, on the happening of either of three different events: 1° that all the three lives become extinct in the year, A having died the second or third: 2° that A and B both fail in the year, A having died last, and that C lives to the end of it: 3° that A and C both fail in the year, A having died last, and that B lives to the end of it. The probabilities of the happening of these several events in the first year, are respectively $\frac{2(a-\overset{\prime}{a}).(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{3abc}$, $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b})\overset{\prime}{c}}{2abc}$, and $\frac{(a-\overset{\prime}{a}).(c-\overset{\prime}{c})\overset{\prime}{b}}{2abc}$: which being added together and multiplied by $s(1+g)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, the sum may be received on the happening of either of eight different events: 1° that all the three lives fail in the year, A having died second or third; 2° that A and B both fail in the year, A having died last, and that C lives to the end of it: 3° that A and C both fail in the year, A having died last, and that B lives to the end of it: 4° that A and B both die in the year, C having died in either of preceding years: 5° that A and C both die in the year, B having died in either of the preceding years: 6° that only A dies in the year, B living to the end of it, and C having died in either of the preceding years: 7° that only A dies in the year, C living to the end of it, and B having died in either of the preceding years: 8° that only A dies in the year, B and C having both failed in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively $\frac{2(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{3abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{c}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{b}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{ac} \times \left(1-\frac{\overset{\prime}{b}}{b}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{c}}{ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$, and $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})}{a}\left(1-\frac{\overset{\prime}{b}}{b}\right).\left(1-\frac{\overset{\prime}{c}}{c}\right)$: which, being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year to the utmost extent of human life: the sum of all which expectations will be the total present value of the given sum, depending on the above contingency.

These several yearly expectations, being reduced to their least terms and arranged under each other, will form ten collateral series: the sum of all which will be found equal to $s\left[\frac{1-\varrho A}{(1+\varrho)}-\mathcal{A}^{BC}\right]$.*

COROLLARY.

§ 265. When all the lives are equal, or of the same age A, the last term in this formula becomes equal to $\frac{1-\varrho AAA}{3(1+\varrho)}$ (agreeably to what has been already said in Prob. XXIX, cor. 2), consequently the whole expression is in this case reduced to $s \times \frac{2-\varrho(3A-AAA)}{3(1+\varrho)}$: that is, equal to the difference between the value of an assurance of the given sum on a single life, and one third of the value of an assurance of the same sum on the three joint lives.

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* The value of the sum depending on the contingency mentioned in this problem is manifestly equal to the *difference* between the value of such sum to be received on the decease of A, and its value depending on the contingency that A shall be the *first* that fails: or, it is equal to the *sum* of the two values found by Prob. XXX and XXXI.

PROBLEM XXXIV.*

§ 266. To determine the present value of a given sum payable on the decease of A, provided he be the *first or last* that fails of three given lives A, B, C.

SOLUTION.

The given sum may be received at the end of the first year on the happening of either of four different events: 1° that all the three lives fail in the year, A having died first or last: 2° that A and B both fail in the year, A having died first, and that C lives to the end of it: 3° that A and C both fail in the year, A having died first, and that B lives to the end of it: 4° that only A dies in the year, and that B and C both live to the end of it. The probabilities of the happening of these several events are respectively

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* Morgan, Prob. 26; and in Phil. Trans. for 1791, Prob. 7, page 265.

$\frac{2(a-\overset{\prime}{a}).(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{3abc}$, $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b})\overset{\prime}{c}}{2abc}$, $\frac{(a-\overset{\prime}{a}).(c-\overset{\prime}{c})\overset{\prime}{b}}{2abc}$, and $\frac{(a-\overset{\prime}{a})\overset{\prime}{b}\overset{\prime}{c}}{abc}$: which, being added together and multiplied by $s(1+\rho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But at the end of the second and following years the given sum may be received on the happening of either of seven different events: 1° that all the lives fail in the year, A having died first or last: 2° that A and B both fail in the year, A having died first, and that C lives to the end of it: 3° that A and C both fail in the year, A having died first, and that B lives to the end of it: 4° that only A dies in the year and that B and C both live to the end of it: 5° that A and B both fail in the year, A having died last, and C having failed in either of the preceding years: 6° that A and C both fail in the year, A having died last, and B having failed in either of the preceding years: 7° that only A dies in the year, B and C having both failed in either of the preceding years. The probabilities of the happening of these several events in the second year, are respectively $\frac{2(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{3abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{c}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{b}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{2ab} \times \left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$, and $\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{a}\left(1-\frac{\overset{\prime}{b}}{b}\right) \times \left(1-\frac{\overset{\prime}{c}}{c}\right)$: which, being added together and multiplied by $s(1+\rho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year to the utmost extent of human life: the sum of all which yearly values will be the total present value of the given sum to be received on the above contingency.

These several yearly expectations, being reduced to their least terms and arranged under each other, will form eighteen collateral series: the sum of all which will be found equal to $s \times \left[\frac{1-\varrho A}{(1+\varrho)} - \mathcal{A}^{B} - \mathcal{A}^{C} + 2\mathcal{A}^{BC}\right]$.*

COROLLARY.

§ 267. When the three lives are equal, or of the same age A, this expression (agreeably to what has been said in the corollary to Prob. XXX) will become equal to $s \times \frac{2-\varrho(3A-3AA+2AAA)}{3(1+\varrho)}$: that is, equal to the difference between the value of an assurance of the given sum on a single life and on two joint lives, added to two thirds of the value of an assurance of the same sum on the three joint lives.

SCHOLIUM.

§ 268. If the present values of the given sum, as

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[*] This formula may be obtained by taking the *sum* of the values deduced from Prob. XXIX and XXXI: or, by taking the *difference* between $s \times \frac{1-\varrho A}{(1+\varrho)}$ and the value deduced from Prob. XXX.

found by Prob. XXXII, XXXIII, and XXXIV, be added together, they will be found equal to twice the present value of the same sum to be received on the decease of A: that is, the sum of those three values will be equal to $2s \times \frac{1-\varrho A}{(1+\varrho)}$.

PROBLEM XXXV.*

§ 269. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *first* that fails of three given lives A, B, C.

SOLUTION.

The given sum may be received at the end of any one year, on the happening of either of six different events: 1° that all the lives fail in the year, A or B having died first: 2° that A and B both die in the year, and that C lives: 3° that A and C both fail in the year, A having died first, and that B lives to the end of it: 4° that B and C both fail in the year, B having died first, and that A lives to the end of it: 5° that only A dies in the year, and that B and C both live to the end of it: 6° that only B dies in the year, and that A and C both live to the end of it. The probabilities of the happening of which several events in the first year are respectively

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* Phil. Trans. for 1791, Prob. 8, page 267.

$\frac{2(a-\overset{\prime}{a}).(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{3abc}$, $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b})\overset{\prime}{c}}{abc}$, $\frac{(a-\overset{\prime}{a}).(c-\overset{\prime}{c})\overset{\prime}{b}}{2abc}$, $\frac{(b-\overset{\prime}{b}).(c-\overset{\prime}{c})\overset{\prime}{a}}{2abc}$, $\frac{(a-\overset{\prime}{a})\overset{\prime}{b}\overset{\prime}{c}}{abc}$, and $\frac{(b-\overset{\prime}{b})\overset{\prime}{a}\overset{\prime}{c}}{abc}$: which, being added together and multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

In like manner may be found the expectation of receiving the sum at the end of the second, third and every subsequent year to the utmost extent of human life: which several yearly values, reduced to their least terms and being arranged under each other as in Prob. XXIX, will form the following series,

$$)\left[\frac{4abc}{abc}-\frac{4\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}-\frac{\overset{\prime}{a}bc}{abc}+\frac{a\overset{\prime}{b}\overset{\prime}{c}}{abc}-\frac{a\overset{\prime}{b}c}{abc}+\frac{\overset{\prime}{a}b\overset{\prime}{c}}{abc}+\frac{2ab\overset{\prime}{c}}{abc}-\frac{2\overset{\prime}{a}\overset{\prime}{b}c}{abc}\right]+$$

$$)^2\left[\frac{4\overset{\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}-\frac{4\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}-\frac{\overset{\prime\prime}{a}\overset{\prime}{b}\overset{\prime}{c}}{abc}+\frac{\overset{\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}-\frac{\overset{\prime}{a}\overset{\prime\prime}{b}\overset{\prime}{c}}{abc}+\frac{\overset{\prime\prime}{a}\overset{\prime}{b}\overset{\prime\prime}{c}}{abc}+\frac{2\overset{\prime}{a}\overset{\prime}{b}\overset{\prime\prime}{c}}{abc}-\frac{2\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime}{c}}{abc}\right]+$$

$$)^3\left[\frac{4\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}-\frac{4\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}-\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}+\frac{\overset{\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}-\frac{\overset{\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime}{c}}{abc}+\frac{\overset{\prime\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}+\frac{2\overset{\prime\prime}{a}\overset{\prime\prime}{b}\overset{\prime\prime\prime}{c}}{abc}-\frac{2\overset{\prime\prime\prime}{a}\overset{\prime\prime\prime}{b}\overset{\prime\prime}{c}}{abc}\right]+$$

c &c &c &c &c,

the sum of all which will be the total present value required.

§ 270. But, if we compare this general series with the one inserted in page 207, it will be found that the several collateral series, of which it is composed, are for the most part the same as those which are there given: and that the *only* difference between them is in the common *multiple*, and in the *sign* prefixed to the fifth and sixth collateral series. Consequently, the sum of these several perpendicular and collateral series may be readily obtained from the process there laid down. For, the first and second collateral series will be thus found equal to $\frac{2s}{3} \times \frac{1-\varrho ABC}{(1+\varrho)}$; the third and fourth, equal to $-\frac{s}{6a}\left[\frac{(1+A'BC)a'}{(1+\varrho)} - A_{\prime}BC.a_{\prime}\right]$; the fifth and sixth, equal to $-\frac{s}{6b}\left[\frac{(1+AB'C)b'}{(1+\varrho)} - AB_{\prime}C.b_{\prime}\right]$; and the seventh and eighth, equal to $\frac{s}{3c}\left[\frac{(1+ABC')c'}{(1+\varrho)} - ABC_{\prime}.c_{\prime}\right]$. Therefore the total value of the general series above given will be equal to s multiplied into

$$\frac{2}{3} \times \frac{1-\varrho ABC}{(1+\varrho)} - \frac{1}{6a}\left[\frac{(1+A'BC)a'}{(1+\varrho)} - A_{\prime}BC.a_{\prime}\right] - \frac{1}{6b}\left[\frac{(1+AB'C)b'}{(1+\varrho)} - AB_{\prime}C.b_{\prime}\right] + \frac{1}{3c}\left[\frac{(1+ABC')c'}{(1+\varrho)} - ABC_{\prime}.c_{\prime}\right].$$

As we shall have occasion to refer again to this formula in some of the following problems, it will be convenient to denote it by a more simple expression; therefore let it be represented by $\mathcal{AB}^{C}$: that is, let $\mathcal{AB}^{C}$ denote the present value of £1 to be received on the above contingency; consequently $s \times \mathcal{AB}^{C}$ will denote the present value of the given sum under the same circumstances.*

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* By comparing this formula with the second formula in Prob. XXIX, cor. 1, it will be seen that $\mathcal{AB}^{C} = \frac{1+ABC}{(1+\varrho)} - C^{AB}$.

COROLLARY.

§ 271. When the three lives are equal, or of the same age A, the last three terms in the above expression destroy each other; and the whole is then reduced to $\frac{2s}{3} \times \frac{1-\varsigma AAA}{(1+\varsigma)}$: or, to two thirds of the present value of the given sum payable on the extinction of the three joint lives.

Observations on Mr. Morgan's Method of investigating this Problem.

§ 272. I cannot omit the present opportunity of recalling the reader's attention once more to the singular and confused manner which Mr. Morgan has adopted, in this case also, for finding the value of the general series in page 239: and I am the more induced to do this, because it appears that, the further Mr. Morgan proceeds in his subject, the more disorder and irregularity seems to run through the whole of his investigations. I would previously, however, intreat the reader's patience whilst I conduct him through this mathematical labyrinth.

The sum of the several expectations of receiving the given sum, on the contingency mentioned in this problem, Mr. Morgan makes equal to the following series

$$\frac{s}{(1+\varrho)}\left[\frac{2(a-a')bc}{3abc}-\frac{(a-a')b'c}{6abc}+\frac{(a-a')bc'}{3abc}+\frac{(a-a')b'c'}{6abc}+\frac{(b-b')c}{2bc}+\frac{(b-b')c'}{2bc}-\frac{(a-a')bc}{2abc}+\frac{(a-a')b'c}{2abc}-\frac{(a-a')bc'}{2abc}+\frac{(a-a')b'c'}{2abc}\right]+$$

$$\frac{s}{(1+\varrho)^2}\left[\frac{2(a'-a'')b'c'}{3abc}-\frac{(a'-a'')b''c'}{6abc}+\frac{(a'-a'')b'c''}{3abc}+\frac{(a'-a'')b''c''}{6abc}+\frac{(b'-b'')c'}{2bc}+\frac{(b'-b'')c''}{2bc}-\frac{(a-a'')b'c'}{2abc}+\frac{(a-a'')b''c'}{2abc}-\frac{(a-a'')b'c''}{2abc}+\frac{(a-a'')b''c''}{2abc}\right]+$$

$$\frac{s}{(1+\varrho)^3}\left[\frac{2(a''-a''')b''c''}{3abc}-\frac{(a''-a''')b'''c''}{6abc}+\frac{(a''-a''')b''c'''}{3abc}+\frac{(a''-a''')b'''c'''}{6abc}+\frac{(b''-b''')c''}{2bc}+\frac{(b''-b''')c'''}{2bc}-\frac{(a-a''')b''c''}{2abc}+\frac{(a-a''')b'''c''}{2abc}-\frac{(a-a''')b''c'''}{2abc}+\frac{(a-a''')b'''c'''}{2abc}\right]+$$

&c &c &c &c &c &c;

the whole value of which, he says, is equal to *s* multiplied into $B_{\prime}C_{\prime}\times\frac{b_{\prime}c_{\prime}}{6bc}-AB_{\prime}C_{\prime}\times\frac{b_{\prime}c_{\prime}}{6bc}+BC_{\prime}\times\frac{c_{\prime}}{3c}-ABC_{\prime}\times\frac{c_{\prime}}{3c}-B_{\prime}C\times\frac{b_{\prime}}{6b}+AB_{\prime}C\times\frac{b_{\prime}}{6b}+\frac{2\varrho(BC-ABC)}{3(1+\varrho)}+\frac{B'C}{(1+\varrho)}\times\frac{b'}{6b}-\frac{AB'C}{(1+\varrho)}\times\frac{b'}{6b}-\frac{BC'}{(1+\varrho)}\times\frac{c'}{3c}+\frac{ABC'}{(1+\varrho)}\times\frac{c'}{3c}-\frac{B'C'}{(1+\varrho)}\times\frac{b'c'}{6bc}+\frac{AB'C'}{(1+\varrho)}\times\frac{b'c'}{6bc}+\frac{\varrho(C-BC)}{2(1+\varrho)}+C_{\prime}\times\frac{c_{\prime}}{2c}-BC_{\prime}\times\frac{c_{\prime}}{2c}-\frac{C'}{(1+\varrho)}\times\frac{c'}{2c}+\frac{BC'}{(1+\varrho)}\times\frac{c'}{2c}$.*

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* These last five terms are the value of Mr. Morgan's Σ, expressed in his own circuitous manner. The reader will, at first, perhaps hesitate in believing that the *whole* of the above complex expression may be denoted by only *seven* of the terms made use of by Mr. Morgan!!! See the proper formula in page 240.

§ 273. Now, the last *six* collateral series in the above general expression may be reduced to the *two* following ones: viz.

$$\frac{s}{(1+\varrho)}\left[\frac{(b-b')a'c}{2abc}+\frac{(b-b')a'c'}{2abc}\right]+$$
$$\frac{s}{(1+\varrho)^2}\left[\frac{(b'-b'')a''c'}{2abc}+\frac{(b'-b'')a''c''}{2abc}\right]+$$
$$\frac{s}{(1+\varrho)^3}\left[\frac{(b''-b''')a'''c''}{2abc}+\frac{(b''-b''')a'''c'''}{2abc}\right]+$$
$$\&c \qquad \&c \qquad \&c,$$

in which case, that expression, instead of being composed of *twenty** different series, might have been reduced to *twelve*. Or, had the whole of these collateral series been reduced to their simplest terms, they would have formed the *very same* expression as that given in page 239; and which consists of only *eight* different terms!!! But, owing to this careless and diffuse mode of treating the subject, it follows that the formula, which denotes the value of the given sum depending on the contingency mentioned in this problem, contains (upon Mr. Morgan's method of solution) no less than twenty different terms:† and when it is considered that, in order to obtain those terms, it is necessary* to sum up between *thirty and forty* different series, the reader is equally struck with surprise and indignation; and is left in doubt of the *motive* which could have induced the author to pursue so anomalous a course.

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* *Each* of the ten collateral series in page 242, evidently forms *two* distinct series when expanded by multiplication; the sum of *both* which it is necessary to find.

† Lest I should be charged with misrepresentation I shall here insert Mr. Morgan's formula in his own characters: viz S into $\frac{x}{3c}\left[\frac{\beta(\mathrm{FK}-\mathrm{AFK})}{2b}+(\mathrm{BK}-\mathrm{ABK})\right]-\frac{\beta(\mathrm{FC}-\mathrm{AFC})}{6b}+\frac{2(r-1).(\mathrm{BC}-\mathrm{ABC})}{3r}+\frac{m(\mathrm{PC}-\mathrm{APC})}{6br}-\frac{d}{3cr}\left[(\mathrm{BT}-\mathrm{ABT})+\frac{m(\mathrm{PT}-\mathrm{APT})}{2b}\right]+\Sigma$ (Σ denoting the value of S on the con-

§ 274. There is, however, still another part of Mr. Morgan's investigation on which I consider myself equally obliged to make some comment. He says that the formula (above alluded to) gives the exact value when C is the oldest of the three lives; " but if A be the oldest, *the symbols must be changed:*" and the value of the given sum he has, in this case, denoted by *another* formula; which (being introduced in a new dress, and in new characters) appears materially different from the former one.† It is, however, (when divested of all its useless and extraneous quantities) precisely the same as that which I have

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tingency of C's surviving B). The other symbols being the same as already explained in the note in page 215. See the original formula in *Phil. Trans.* for 1791, page 267; or in Price's *Obs. on Rev. Pay.* Note (P), Problem VIII.

* I mean upon Mr. Morgan's principles, as explained in page 217.

† Take Mr. Morgan's own words. " The general rule expressing the value of the reversion will be $=$ S into $\frac{(r-1).(V-AB)}{r}$

$$-\frac{\alpha}{3a}\left[\frac{\beta(HF-HFC)}{b}+\frac{HB-HBC}{2}\right]-\frac{\beta(AF-AFC)}{6b}+$$

$$\frac{2(r-1).(AB-ABC)}{3r}+\frac{m(AP-APC)}{6br}+\frac{s}{3ar}\left[\frac{(BN-BNC)}{2}+\right.$$

given in page 240, except as to the substitution of $-\frac{1}{3ab}\left[\frac{(1+\overset{\prime}{A}\overset{\prime}{B}C)\overset{\prime}{a}\overset{\prime}{b}}{(1+\varrho)} - \underset{\prime}{A}\underset{\prime}{B}C.\underset{\prime}{a}\underset{\prime}{b}\right]$ for its equal $+\frac{1}{3c}\left[\frac{(1+AB\overset{\prime}{C})\overset{\prime}{c}}{(1+\varrho)} - AB\underset{\prime}{C}.\underset{\prime}{c}\right]$; agreeably to the principles explained in the note in page 210.

§ 275. It surely cannot be unknown to Mr. Morgan that the value of the given sum does not in this case depend upon the *seniority* of the lives concerned.* For, let the ages of A, B, and C be what they may, and let him change his symbols as often as he pleases, he must still come to the *same* general series, at last, as that given in page 239: the sum of which may be expressed in three different ways, according to the method of summing up the several collateral series of which it is composed.† And *either* of these formulæ will denote the value of the given sum, whether A, B, or C be the oldest of the three lives.

$\left.\frac{m(PN-PNC)}{b}\right]$." Where V denotes the perpetuity; and the remaining letters, the same quantities as already explained in the note in page 221. See the original formula in *Phil. Trans.* for 1791, page 268; and in Price's *Obs. on Rev. Pay.* Note (P), Problem VIII.

* This observation will apply to the whole of the Problems inserted by Mr. Morgan in the *Phil. Trans.* for 1788, 1789 and 1791. See the note in page 222.

† See what has been already said on this subject in page 222, and in the note to page 220.

These observations are the more necessary to be attended to, because the solution to the present problem is of use in enabling us to determine the value of sums depending on the contingencies mentioned in several of the subsequent problems. Consequently, the more simple we render the present formula, the more easily we may obtain the solution to those problems. This motive, therefore, will be a sufficient apology for the present digression.

PROBLEM XXXVI.*

§ 276. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *second* that fails of three given lives A, B, C.

SOLUTION.

The given sum may be received at the end of the first year, on the happening of either of four different events: 1° that all the three lives fail in that year, A or B having died second: 2° that A and B both die in that year, and C lives to the end of it: 3° that A and C both fail in that year, A having died last, and that B lives to the end of it: 4° that B and C both fail in that year, B having died last, and that A lives to the end of it. The probabilities of the happening of these several events are respectively $\frac{2(a-a').(b-b').(c-c')}{3abc}$, $\frac{(a-a').(b-b')c'}{abc}$, $\frac{(a-a').(c-c')b'}{2abc}$, and $\frac{(b-b').(c-c')a'}{2abc}$: which being added together, and multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

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* Phil. Trans. for 1791, page 269.

But at the end of the second and following years, the sum may be received on the happening of either of eleven different events: 1° that all the lives fail in the year, A or B having died second: 2° that A and B both die in the year, and that C lives to the end of it: 3° that A and C both fail in the year, A having died last, and that B lives to the end of it: 4° that B and C both fail in the year, B having died last, and that A lives to the end of it: 5° that A and B both die in the year, C having died in either of the preceding years: 6° that A and C both fail in the year, A having died last, and B having died in either of the preceding years: 7° that B and C both fail in the year, B having died last, and A having died in either of the preceding years: 8° that only A dies in the year, B living to the end of it, and C having died in either of the preceding years: 9° that only A dies in the year, C living to the end of it, and B having died in either of the preceding years: 10° that only B dies in the year, A living to the end of it, and C having died in either of the preceding years: 11° that only B dies in the year, C living to the end of it, and A having died in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively $\frac{2(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{3abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{c}}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{b}}{2abc}$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{a}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{ab}\times\left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2bc}\left(1-\frac{\overset{\prime}{a}}{a}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{c}}{ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{a}}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{c}}{bc}\left(1-\frac{\overset{\prime}{a}}{a}\right)$: which being added together, and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner may be found the expectation of receiving the sum at the end of the third and every subsequent year to the utmost extent of human life: the sum of all which yearly values will be the total present value of the given sum, to be received on the above contingency.

Now these several yearly expectations, being reduced to their lowest terms and arranged under each other, will form eighteen collateral series: the sum of all which will be found equal to $s\left[\frac{1-\varrho AB}{(1+\varrho)}+A^{C}+B^{C}-2.AB^{C}\right]$.

COROLLARY.

§ 277. When all the lives are equal, or of the same age A, this expression (agreeably to what has been said in the corollary to Prob. XXX) will become equal to $\frac{2s}{3}\times\frac{1-\varrho(3AA-2AAA)}{(1+\varrho)}$: or, to two thirds of the present value of the given sum payable on the extinction of any two out of three given lives.

PROBLEM XXXVII.*

§ 278. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *last* that fails of three given lives A, B, C.

SOLUTION.

The given sum can be received at the end of the first year, on the happening of one event only; viz. the extinction of all the lives in that year, C having been the first or second that failed. The probability of this event is $\frac{2(a-\overset{\prime}{a}).(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{3abc}$; which being multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of six different events: 1° that all the three lives fail in the year, C having died first or second: 2° that only A and B fail in the year, C having died in either of the preceding years: 3° that A and C both fail in the year, A having died last, and B having died in either of the preceding years: 4° that B and C both fail in the year, B having died last, and A having died in either of the preceding years: 5° that only A dies in the year, B and C having both failed in either of the preceding years: 6° that only B dies in the year, A and C having both failed in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively $\frac{2(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{3abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2ac}\times\left(1-\frac{\overset{\prime}{b}}{b}\right)$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2bc}\left(1-\frac{\overset{\prime}{a}}{a}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})}{a}\left(1-\frac{\overset{\prime}{b}}{b}\right)\times\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b})}{b}\left(1-\frac{\overset{\prime}{a}}{a}\right).\left(1-\frac{\overset{\prime}{c}}{c}\right)$: which being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third year. And so on for all the subsequent years to the utmost extent of human life: the sum of all which yearly values will be the total present value of the given sum, payable on the above contingency.

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* Phil. Trans. for 1791, Prob. 10, page 272.

These several yearly expectations, being reduced to their lowest terms and arranged under each other, will form twenty-two collateral series: the sum of all which will be found equal to $s\left[\frac{1-\varrho(A+B-AB)}{(1+\varrho)}-\mathcal{A}^{C}-\mathcal{B}^{C}+\mathcal{AB}^{C}\right]$.

COROLLARY.

§ 279. When the three lives are equal, or of the same age A, this expression will become equal to $\frac{2s}{3} \times \frac{1-\varrho(3A-3AA+AAA)}{(1+\varrho)}$: or, two thirds of the present value of the given sum, payable on the extinction of the longest of the three lives.

SCHOLIUM.

§ 280. If the present values of the given sum, as found by Prob. XXXV, XXXVI, and XXXVII, be added together, they will be found equal to the present value of the same sum to be received on the decease of A, *added to* the present value of the same sum to be received on the decease of B: that is, the sum of such three values will be equal to $s\left[\frac{1-\varrho A}{(1+\varrho)}+\frac{1-\varrho B}{(1+\varrho)}\right]$.

PROBLEM XXXVIII.*

§ 281. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *first* or *second* that fails of three given lives A, B, C.

SOLUTION.

It is evident that in this case the payment of the given sum at the end of any one year depends wholly on the extinction of the joint lives AB in that year, independent of C: its present value therefore will, by Prob. XXII, be in all cases equal to $s \times \frac{1-\varrho AB}{(1+\varrho)}$.

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* Phil. Trans. for 1800, Prob. 1, page 22.

SCHOLIUM.

§ 282. In the preceding problems I have deduced *correct* expressions for the value of reversionary sums depending on the several contingencies therein mentioned: but the remaining problems, for the most part, involve a contingency for which it is very difficult to find such an expression as will denote the true value of the same, and be likewise fit for general use. The contingency, to which I allude, is the probability that one life in particular will die before or after another, during any given period of their joint lives. This subject has been already discussed in the fifth chapter, where it is applied to the method of determining the present value of certain reversionary *annuities:* and I now come to consider it again in its application to the method of determining the present value of reversionary *sums.* In the investigation of the following problems, the contingencies above mentioned will be expressed in italics. And since, by means of the two Lemmata in Chapter V, we may obtain a more convenient expression for the expectations of receiving the given sum after the extinction of the oldest life involved, I shall divide the investigation into two distinct parts; the first of which will denote the value of all the expectations for the first *n* years,* or during the continuance of the oldest life involved; and the second will denote the value of all the expectations after that period.

PROBLEM XXXIX.†

§ 283. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *second* or *third* that fails of three given lives A, B, C.

SOLUTION.

The given sum may be received at the end of the first year on the happening of either of four different events: 1° that all the lives fail in that year: 2° that A and B fail in that year, and that C lives to the end of it: 3° that A dies after C in that year, and that B lives to the end of it: 4° that B dies after C in that year, and that A lives to the end of it. The probabilities of the happening of these several events are respectively $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{abc}$, $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b})\overset{\prime}{c}}{abc}$, $\frac{(a-\overset{\prime}{a}).(c-\overset{\prime}{c})\overset{\prime}{b}}{2abc}$, and $\frac{(b-\overset{\prime}{b}).(c-\overset{\prime}{c})\overset{\prime}{a}}{2abc}$: which, being added together and multiplied by $s(1+\rho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

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* I shall here observe, once for all, that I take *n* to denote the number of years between the age of the oldest life and that age in the table of observations when human life becomes extinct: consequently the value of *n* will *vary* in *each* problem according to the seniority of the lives concerned.

† Phil. Trans. for 1800, Prob. 2, page 23.

But in the second and following years, during the existence of the oldest life, the given sum may be received on the happening of either of thirteen different events: 1° that all the lives fail in the year: 2° that A and B fail in the year, and that C lives to the end of it: 3° that A and C both fail in the year, A having died last, and that B lives to the end of it: 4° that B and C both fail in the year, B having died last, and that A lives to the end of it: 5° that A and B die in the year, C having failed in either of the preceding years: 6° that A and C die in the year, B having failed in either of the preceding years: 7° that B and C die in the year, A having failed in either of the preceding years: 8° that only A dies in the year, B living to the end of it, and C having died in either of the preceding years: 9° that only A dies in the year, C living to the end of it, and B having died in either of the preceding years: 10° that only B dies in the year, A living to the end of it, and C having died in either of the preceding years: 11° that only B dies in the year, C living to the end of it, and A having died in either of the preceding years: 12° that only A dies in the year, and *that B and C both fail in either of the preceding years, B having died first:* 13° that only B dies in the year, and *that A and C both fail in either of the preceding years, A having died first.* The probabilities of the happening of these several events in the second year, are respectively $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{c}}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{b}}{2abc}$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{a}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{ab} \times \left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{bc}\left(1-\frac{\overset{\prime}{a}}{a}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$ $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{c}}{ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{a}}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{c}}{bc}\left(1-\frac{\overset{\prime}{a}}{a}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})}{2a}\left(1-\frac{\overset{\prime}{b}}{b}\right)\cdot\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and $\frac{\overset{\prime}{b}-\overset{\prime\prime}{b}}{2b} \times \left(1-\frac{\overset{\prime}{a}}{c}\right)\cdot\left(1-\frac{\overset{\prime}{c}}{c}\right)$:* which being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year: and, if these several annual expectations be reduced to their lowest terms and arranged under each other, they will form sixteen collateral series; the sum of which, continued to the utmost extent of human life, would be equal to *s* multiplied into $\frac{1-\varrho A}{2(1+\varrho)} + \frac{1-\varrho B}{2(1+\varrho)} - AB + \frac{1+\overset{\prime}{A}B}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a} + \frac{1+A\overset{\prime}{B}}{2(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{b} + \frac{1+AC}{2(1+\varrho)} - \frac{1+\overset{\prime}{A}C}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a} + \frac{1+BC}{2(1+\varrho)} - \frac{1+\overset{\prime}{B}C}{2(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{b} - \frac{1-\varrho ABC}{(1+\varrho)} - \underset{\prime}{A}BC \times \frac{\underset{\prime}{a}}{2a} - A\underset{\prime}{B}C\cdot\frac{\underset{\prime}{b}}{2b} + AB\underset{\prime}{C}\cdot\frac{\underset{\prime}{c}}{c}$. But this expression, independent of the common multiple *s*, may be reduced to $\frac{2-\varrho(A+B-2ABC)+AC+BC}{2(1+\varrho)} - AB + \frac{1}{2a}\left[\frac{(\overset{\prime}{A}B-\overset{\prime}{A}C)\overset{\prime}{a}}{(1+\rho)} - \underset{\prime}{A}BC.\underset{\prime}{a}\right] + \frac{1}{2b}\left[\frac{(A\overset{\prime}{B}-\overset{\prime}{B}C)\overset{\prime}{b}}{(1+\varrho)} - A\underset{\prime}{B}C.\underset{\prime}{b}\right] + AB\underset{\prime}{C}\cdot\frac{\underset{\prime}{c}}{c}$: and which, for the sake of a more convenient reference, I shall denote by 𝔻.

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* In these, and all similar cases, throughout the remaining part of this chapter, I suppose it to be an *equal* chance which of the two lives dies before or after the other during the probable term of their joint existence. Now, though this is not strictly correct, (and I have on that account pointed out, in page 139, a method of finding an approximate value of such chance in order to determine the value of contingent *Annuities*,) yet in the case of *Assurances* the contingencies are so involved that no material error will probably arise in any practical cases by thus indiscriminately supposing the chances to be even. The resulting formulæ are hereby rendered more simple and easy, and are sufficiently accurate for general use.

§ 284. Now, since only part of the several collateral series above mentioned (and which series are represented by the quantities here given) is to be continued to the utmost extent of human life, and as that part will depend on the seniority of the lives concerned; it will be necessary to divide the subsequent investigation of the problem into three distinct cases, according to the seniority of the three lives. For, in each case those several collateral *series* must be continued only *during the existence of the oldest of such lives:* because, *after* that period, we may obtain a more correct value of all the subsequent expectations, by means of the first Lemma in the fifth chapter.

§ 285. *Case* 1. Let A be the oldest of the three lives. In this case, all those collateral series in which the life A is involved must of course be continued to the utmost extent of human life: but all those series in which the life A is not involved must be continued for n terms only. Consequently, the quantities $\frac{1-\varrho B}{2(1+\varrho)}$, $\frac{1+BC}{2(1+\varrho)}$, and $\frac{1+\acute{B}C}{2(1+\varrho)}\cdot\frac{\acute{b}}{b}$ will respectively become $\left[\frac{1-\varrho B}{2(1+\varrho)}-\frac{1-\varrho B^{\circ}}{2(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n}\right]$, $\left[\frac{1+BC}{2(1+\varrho)}-\frac{1+B^{\circ}C^{\circ}}{2(1+\varrho)}\cdot\frac{\beta\gamma}{bc}(1+\varrho)^{-n}\right]$ and $\left[\frac{1+\acute{B}C}{2(1+\varrho)}\cdot\frac{\acute{b}}{b}-\frac{1+\acute{B}^{\circ}C^{\circ}}{2(1+\varrho)}\times\frac{\acute{\beta}\gamma}{bc}(1+\varrho)^{-n}\right]$:* whence, the sum of the first n terms

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* See the method of proceeding in these cases explained in Prob. XXII, cor. 2, and Prob. XXVII, cor. 4. I shall, however, here take the opportunity of noticing an error on this subject, into which Mr. Morgan has inadvertently fallen; and which pervades the two papers inserted by him in the 84th and 90th volumes of the *Philosophical Transactions.* In pointing out the method of finding the value of the first n terms of the several series above alluded to, in which the oldest life is not involved, he directs us to convert the quantities, expressing the value of annuities on the *whole* life, into similar expressions for the given *term.* This however is certainly incorrect: for, it is not the *annuities* on those lives which are to be continued for n years, but the *series* from which those annuities arise; and which will in most cases make the value of the annuities, resulting therefrom, equal to a term of $n-1$ years only.

Thus, in the investigation of the present problem in the *Philosophical Transactions* (for 1800, part I, page 25) he says that, when B is the oldest of the three lives, the annuity on the joint lives AC must be continued for x years: whereas it should be continued of the several collateral series above mentioned (that is, the sum of all the expectations for the first n years) will become equal to s multiplied into $\mathbb{D} - \frac{1-\varrho B'}{2(1+\varrho)} \times \frac{\beta}{b}(1+\varrho)^{-n} - \frac{1+B'C'}{2(1+\varrho)} \cdot \frac{\beta\gamma}{bc}(1+\varrho)^{-n} + \frac{1+\acute{B}'C^{\circ}}{2(1+\varrho)} \times \frac{\acute{\beta}\gamma}{bc}(1+\varrho)^{-n}$.

But, after the decease of A, the expectations, arising from the several contingencies on which the sum depends, may be more correctly expressed for all the subsequent years by means of the first Lemma in Chapter V. For, since the chance of receiving the sum at the end of any one of those years depends on B's dying

for $x-1$ years only; as will readily appear, by attending to the steps of the process. The annuity on the life A, he also says, should be continued for x years: whereas neither x nor $x-1$ will give the correct value in his formula; for, the true value which ought to be *there* substituted for A is $A' + \frac{1}{r-1} \times \frac{q}{ar^x}$. As this error runs through the *whole* of the investigations inserted by Mr. Morgan in the two papers above mentioned, I have thought it necessary to advert to it in this place, in order to explain any difference which might be observed between his formulæ and those which are here given.

I cannot dismiss this subject without censuring the careless manner in which Mr. Morgan's papers in the *Philosophical Transactions* have been printed. Were a proof of this charge required, I could not give a more convincing one than the two formulæ in the page above referred to, and which are too erroneous to be of any public utility. I at first imagined that these mistakes might be *errors of the press*; but, when I observe them faithfully copied into the last edition of Dr. Price's treatise, I can be at no loss in attributing them to their proper source.

in the year, and on A having died before C in either of the preceding years (the probability of which latter contingency is, by the Lemma, denoted by ϖ[*]) it follows that the sum of the expectations for those years, continued to the utmost extent of human life, will be equal to $s.\varpi \times \left[\frac{\beta - \beta'}{b(1+\varrho)^{n+1}} + \frac{\beta' - \beta''}{b(1+\varrho)^{n+2}} + \frac{\beta'' - \beta'''}{b(1+\varrho)^{n+3}} + \&c\right]$: an expression which, by Problem

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[*] Mr. Morgan has given two separate tables for determining the probabilities of survivorship between two lives: the first (which is inserted in *Phil. Trans.* for 1788, page 337) shows the probability of one life dying *before* another; and the second (which is inserted in *Phil. Trans.* for 1794, page 229) shows the probability of one life dying *after* the other. They are both given in the last edition of Dr. Price's *Obs. on Rev. Pay.* vol. 1, page 406, &c.

These tables are deduced from the principles laid down in the first and second Lemma in the fifth Chapter of the present work. The *first* table, here alluded to, will occasionally be found useful in determining the values of ϖ and φ, which are frequently introduced in the ensuing problems. But as that table is very limited in its extent, I have (in the note in page 114) laid down a method of approximating to those probabilities; which in most probable cases will be found sufficiently correct.

Mr. Morgan might have saved himself the trouble of calculating the *second* table above alluded to; since its application to any practical cases may always be avoided: and moreover, the values in that table are, in my opinion, incorrectly deduced. It is certain that the solutions to the ensuing problems may be more simply expressed by referring *only* to the first table; as any person will readily discover by comparing the formulæ in the present work with those given by Mr. Morgan. See what has been already observed on this subject in the note in page 118.

XXII, cor. 1, is equal to $s.\varpi \times \frac{1-\varrho B^{\circ}}{(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n}$.* Consequently this value, added to the sum of the first n terms of the several collateral series above mentioned, will express the total present value of the given sum in this case required; and which will be found equal to s multiplied into

$$\mathbb{D} - \frac{(\frac{1}{2}-\varpi)\times(1-\varrho B^{\circ})\beta + \frac{\gamma}{2c}\left[(1+B^{\circ}C^{\circ})\beta - (1+\acute{B}^{\circ}C^{\circ})\acute{\beta}\right]}{b(1+\varrho)^{n+1}}.$$

§ 286. *Case* 2. Let B be the oldest of the three lives. In this case all those collateral series in which the life B is involved must be continued to the utmost extent of human life: but all those series in which the life B is not involved must be continued for n terms only. Consequently, the quantities $\frac{1-\varrho A}{2(1+\varrho)}$, $\frac{1+AC}{2(1+\varrho)}$, and $\frac{1+\acute{A}C}{2(1+\varrho)}\cdot\frac{\acute{a}}{a}$ will respectively become equal to $\left[\frac{1-\varrho A}{2(1+\varrho)} - \frac{1-\varrho A^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}\right]$, $\left[\frac{1+AC}{2(1+\varrho)} - \frac{1+A^{\circ}C^{\circ}}{2(1+\varrho)} \times \frac{\alpha\gamma}{ac}(1+\varrho)^{-n}\right]$ & $\left[\frac{1+\acute{A}C}{2(1+\varrho)}\cdot\frac{\acute{a}}{a} - \frac{1+\acute{A}^{\circ}C^{\circ}}{2(1+\varrho)}\cdot\frac{\acute{\alpha}\gamma}{ac}(1+\varrho)^{-n}\right]$: whence, the sum of the first n terms of the several collateral series above mentioned will be equal to s multiplied into $\mathbb{D} - \frac{1-\varrho A^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n} - \frac{1+A^{\circ}C^{\circ}}{2(1+\varrho)} \times \frac{\alpha\gamma}{ac}(1+\varrho)^{-n} + \frac{1+\acute{A}^{\circ}C^{\circ}}{2(1+\varrho)}\cdot\frac{\acute{\alpha}\gamma}{ac}(1+\varrho)^{-n}$.

But after the decease of B, the expectations, arising from the several contingencies on which the sum depends, may be more correctly expressed for all the subsequent years by means of the first Lemma in Chapter V. For, since the chance of receiving the sum at the end of any one of those years depends on A's dying in the year, and on B having died before C in either of the preceding years (the probability of which latter contingency is, by the Lemma, denoted by ϕ) it follows, from what has been said in the last case, that the sum of all the expectations for those years, continued to the utmost extent of human life, will be equal to $s.\phi \times \frac{1-\varrho A'}{(1+\varrho)} \times \frac{\alpha}{a}(1+\varrho)^{-n}$. Consequently this value, added to the sum of the first n terms of the several collateral series above mentioned, will express the total present value of the given sum in this case required; and which will be found equal to s multiplied into

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* See the note in page 147.

$$\mathbb{D} - \frac{(\frac{1}{2}-\phi) \times (1-\varrho A')\alpha + \frac{\gamma}{2c}\left[(1+A'C')\alpha - (1+\acute{A}'C')\alpha'\right]}{a(1+\varrho)^{n+1}}.$$

§ 287. *Case* 3. Let C be the oldest of the three lives. In this case, all those collateral series in which the life C is involved must be continued to the utmost extent of human life: but all those series in which the life C is not involved must be continued for n terms only.* Consequently the quantities $\frac{1-\varrho A}{2(1+\varrho)}$, $\frac{1-\varrho B}{2(1+\varrho)}$,

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* Mr. Morgan, in his Investigation of these several cases, has followed his general practice of *changing the symbols* in the *last*

AB, $\frac{1+\overset{\prime}{A}B}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a}$, and $\frac{1+A\overset{\prime}{B}}{2(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{b}$ will respectively become equal to $\left[\frac{1-\varrho A}{2(1+\varrho)}-\frac{1-\varrho A^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}\right]$, $\left[\frac{1-\varrho B}{2(1+\varrho)}-\frac{1-\varrho B'}{2(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n}\right]$, $\left[AB-A^{\circ}B^{\circ}\cdot\frac{\alpha\beta}{ab}(1+\varrho)^{-n}\right]$, $\left[\frac{1+\overset{\prime}{A}B}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a}-\frac{1+\overset{\prime}{A}B^{\circ}}{2(1+\varrho)}\cdot\frac{\overset{\prime}{\alpha}\beta}{ab}(1+\varrho)^{-n}\right]$ and $\left[\frac{1+A\overset{\prime}{B}}{2(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{b}-\frac{1+A^{\circ}\overset{\prime}{B}{}^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha\overset{\prime}{\beta}}{ab}(1+\varrho)^{-n}\right]$: whence, the sum of the first n terms of the several collateral series above mentioned will be equal to s multiplied into $\mathbb{D}-\frac{1-\varrho A^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}-\frac{1-\varrho B^{\circ}}{2(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n}+A^{\circ}B^{\circ}\times\frac{\alpha\beta}{ab}(1+\varrho)^{-n}-\frac{1+\overset{\prime}{A}{}^{\circ}B^{\circ}}{2(1+\varrho)}\cdot\frac{\overset{\prime}{\alpha}\beta}{ab}(1+\varrho)^{-n}-\frac{1+A^{\circ}\overset{\prime}{B}{}^{\circ}}{2(1+\varrho)}\times\frac{\alpha\overset{\prime}{\beta}}{ab}(1+\varrho)^{-n}$.

case, and of introducing *another* formula for the solution thereof. But these new formulæ will always be found to be the *same* as those which he has before inserted, *disguised under different characters.* They arise from the *different* modes of summing up the *same* series: agreeably to what I have before remarked in the notes in page 186 and 210.

In the notation adopted by Mr. Morgan the similarity of the two formulæ is not immediately recognized: and it is possible that he himself was not aware of it. Yet it is singular that he should ever have thought it necessary or useful to recur to *another* expression for the value of the different series, when he had already obtained a formula for that purpose, which (I presume) he must well know was a *general* one. This remark extends to most of the subsequent problems in the present chapter: and will show that, by such acts of supererogation, he has rendered the subject unnecessarily tedious and confused. See the note to the first case of Prob. XLV.

But after the decease of C, the chance of receiving the given sum at the end of any subsequent year will depend on the happening of either of three events: 1° that *either** A or B dies in the year: 2° that only A dies in the year, B having died before C in either of the preceding years: 3° that only B dies in the year, A having died before C in either of the preceding years. The sum of all the expectations of receiving the given sum on the first contingency, for every subsequent year to the utmost extent of human life, is, by Prob. XXII, cor. 1, equal to $s \times \frac{1-\varrho A^{\circ}B^{\circ}}{(1+\varrho)} \times \frac{\alpha\beta}{ab}(1+\varrho)^{-n}$;† and the sum of all the expectations on the other two contingencies is, as in the two preceding cases, equal to $s.\varphi \times \frac{1-\varrho A^{\circ}}{(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n} + s.\varpi \times \frac{1-\varrho B^{\circ}}{(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n}$. Consequently, these values, added to the sum of the first n terms of the several collateral series above mentioned, will express the total present value of the given sum in this case required; and which will be found equal to s multiplied into

$$\mathbb{D} - \frac{(\frac{1}{2}-\varphi)\times(1-\varrho A^{\circ})\alpha - \frac{\beta}{2b}\left[(1+A^{\circ}B^{\circ})\alpha - (1+A'^{\circ}B^{\circ})\alpha'\right]}{a(1+\varrho)^{n+1}}$$

$$- \frac{(\frac{1}{2}-\varpi)\times(1-\varrho B^{\circ})\beta - \frac{\alpha}{2a}\left[(1+A^{\circ}B^{\circ})\beta - (1+A^{\circ}B'^{\circ})\beta'\right]}{b(1+\varrho)^{n+1}}.$$

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* Mr. Morgan says (*Phil. Trans.* for 1800, page 26) that this chance depends on *both* the lives becoming extinct within the year; which is evidently erroneous.

† See the note in page 147.

COROLLARY.

§ 288. When the three lives are equal, or all of the same age A, the last three quantities in the formula denoted by $\mathbb{D}$ vanish entirely; and the general expression for this case will become equal to $s \times \frac{1-\varrho(A+AA-AAA)}{(1+\varrho)}$.

SCHOLIUM.

§ 289. As the expressions, deduced from the method of investigation pursued in these three cases, will be frequently referred to in the subsequent problems, it will be convenient to denote them by a more simple character. Let us therefore make

$$w = \frac{(\frac{1}{2}-\varpi).(1-\varrho B^{\circ})\beta + \frac{\gamma}{2c}\left[(1+B^{\circ}C^{\circ})\beta-(1+\acute{B}^{\circ}\acute{C}^{\circ})\acute{\beta}\right]}{b(1+\varrho)^{n+1}}$$

$$x = \frac{(\frac{1}{2}-\varphi).(1-\varrho A^{\circ})\alpha + \frac{\gamma}{2c}\left[(1+A^{\circ}C^{\circ})\alpha-(1+\acute{A}^{\circ}\acute{C}^{\circ})\acute{\alpha}\right]}{a(1+\varrho)^{n+1}}$$

$$y = \frac{(\frac{1}{2}-\varpi).(1-\varrho B^{\circ})\beta - \frac{\alpha}{2a}\left[(1+A^{\circ}B^{\circ})\beta-(1+\acute{A}^{\circ}\acute{B}^{\circ})\acute{\beta}\right]}{b(1+\varrho)^{n+1}}$$

$$z = \frac{(\frac{1}{2}-\varphi).(1-\varrho A^{\circ})\alpha - \frac{\beta}{2c}\left[(1+A^{\circ}B^{\circ})\alpha-(1+\acute{A}^{\circ}\acute{B}^{\circ})\acute{\alpha}\right]}{a(1+\varrho)^{n+1}}$$

Then will the present value of the given sum depending on the contingency mentioned in this problem be respectively expressed by these several formulæ,

$$s\left[\mathbb{D}-w\right]$$

$$s\left[\mathbb{D}-x\right]$$

$$s\left[\mathbb{D}-y-z\right];$$

according as A, B, or C is the oldest of the three lives.

PROBLEM XL.*

§ 290. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *first* or *last* that fails of three given lives A, B, C.

SOLUTION.

The payment of the given sum at the end of the first year will depend on the happening of either of six different events: 1° that all the lives fail in the year: 2° that A and B fail in the year, and that C lives to the end of it: 3° that A and C fail in the year, A having died first, and that B lives to the end of it: 4° that B and C fail in the year, B having died first, and that A lives to the end of it: 5° that only A dies in the year, and that both B and C live to the end of it: 6° that only B dies in the year, and that both A and C live to the end of it. The probabilities of the happening of these several events in the first year are respectively $\frac{(a-a').(b-b').(c-c')}{abc}$, $\frac{(a-a').(b-b')c'}{abc}$, $\frac{(a-a').(c-c')b'}{2abc}$, $\frac{(b-b').(c-c')a'}{2abc}$, $\frac{(a-a')b'c'}{abc}$ and $\frac{(b-b')a'c'}{abc}$: which being added together and multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

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* Phil. Trans. for 1800, Prob. 3, page 28.

But, in the second and following years, the given sum may be received on the happening of either of nine different events: viz, in addition to the six just enumerated, 7° that A and B die in the year, C having failed in either of the preceding years: 8° that only A fails in the year, and *that B and C both fail in either of the preceding years, C having died first*: 9° that only B fails in the year, and *that A and C both fail in either of the preceding years, C having died first.* The probabilities of the happening of these several events in the second year are respectively $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{c}}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{b}}{2abc}$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{a}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{a}\overset{\prime\prime}{c}}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{2a}\left(1-\frac{\overset{\prime}{b}}{b}\right)\cdot\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and $\frac{\overset{\prime}{b}-\overset{\prime\prime}{b}}{2b}\left(1-\frac{\overset{\prime}{a}}{a}\right)\cdot\left(1-\frac{\overset{\prime}{c}}{c}\right)$: which, being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner, we may find the expectation of receiving the sum at the end of the third and every subsequent year: and, if these several annual expectations be reduced to their lowest terms, and arranged under each other, they will form sixteen collateral series; the sum of which, continued to the utmost extent of human life, would be equal to s multiplied into $\frac{1-\varrho A}{2(1+\varrho)}+\frac{1-\varrho B}{2(1+\varrho)}+AB-\frac{1+\overset{\prime}{A}B}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a}$ $-\frac{1+A\overset{\prime}{B}}{2(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{2b}-\frac{1+AC}{2(1+\varrho)}+\frac{1+\overset{\prime}{A}C}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a}-\frac{1+BC}{2(1+\varrho)}+$

$\frac{1+\overset{\prime}{B}C}{2(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{b}+\frac{1-\varrho ABC}{(1+\varrho)}+\underset{\prime}{A}BC.\frac{\overset{a}{\prime}}{2a}+A\underset{\prime}{B}C.\frac{\overset{b}{\prime}}{2b}-$ $AB\underset{\prime}{C}.\frac{\overset{c}{\prime}}{c}$: an expression which, independent of the common multiple *s*, may be reduced to the formula

$$\frac{2-\varrho(A+B+2ABC)-AC-BC}{2(1+\varrho)}+AB-\frac{1}{2a}\left[\frac{(\overset{\prime}{A}B-\overset{\prime}{A}C)\overset{\prime}{a}}{(1+\varrho)}-\underset{\prime}{A}BC.\underset{\prime}{a}\right]-\frac{1}{2b}\left[\frac{(A\overset{\prime}{B}-\overset{\prime}{B}C)\overset{\prime}{b}}{(1+\varrho)}-A\underset{\prime}{B}C.\underset{\prime}{b}\right]-AB\underset{\prime}{C}.\frac{\overset{c}{\prime}}{2c};$$

and which, for the sake of a more convenient reference I shall denote by $\mathbb{E}$.

§ 291. *Case* 1. Let A be the oldest of the three lives. In this case, the sum of the first *n* terms of the several collateral series above mentioned will, by pursuing the same steps as in the first case of the preceding problem, be equal to $\mathbb{E}-\frac{1-\varrho B'}{2(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n}+\frac{1+B'C'}{2(1+\varrho)}\cdot\frac{\beta\gamma}{bc}(1+\varrho)^{-n}-\frac{1+\overset{\prime}{B}'C'}{2(1+\varrho)}\cdot\frac{\overset{\prime}{\beta}\gamma}{bc}(1+\varrho)^{-n}$. After the decease of A, however, the expectations arising from these contingencies may be more correctly expressed for all the subsequent years by means of the second Lemma in the fifth chapter: for, since the chance of receiving the sum at the end of any one of those years depends on B's dying in the year and on A having died after C in either of the preceding years (the probability of which latter contingency is, by the Lemma, denoted by $1-\varpi$), we shall find that the sum of the expectations for those years, continued to the utmost extent of human life, will be equal to $s.(1-\varpi)\times\frac{1-\varrho B'}{(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n}$. Consequently this value, added to the sum of the first n terms of the several collateral series above mentioned, will make the total present value of the given sum to be equal to $s\left[\mathbb{E}+w\right]$.

§ 292. *Case* 2. Let B be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned will, by pursuing the same steps as in the second case to the preceding problem, be now equal to $\mathbb{E}-\frac{1-\varrho A'}{2(1+\varrho)}\times\frac{\alpha}{a}(1+\varrho)^{-n}+\frac{1+A'C'}{2(1+\varrho)}\cdot\frac{\alpha\gamma'}{ac}(1+\varrho)^{-n}-\frac{1+\acute{A}'C'}{2(1+\varrho)}\times\frac{\acute{\alpha}\gamma'}{ac}(1+\varrho)^{-n}$. After the decease of B, however, the expectation arising from these contingencies may be more correctly expressed for all the subsequent years by means of the Lemma above mentioned: for, since the chance of receiving the sum at the end of any one of those years depends on A's dying in the year, and on B having died after C in either of the preceding years, (the probability of which is, by the Lemma, denoted by $1-\phi$) we shall find that the sum of the expectations for those years, continued to the utmost extent of human life, will be equal to $s.(1-\phi)\times\frac{1-\varrho A'}{(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}$. Consequently, this value added to the sum of the first n terms as above found, will express the total present value of the given sum in this case required. Whence, such present value will be equal to $s\left[\mathbb{E}+x\right]$.

§ 293. *Case* 3. Let C be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned, will (agreeably to the method pursued in the third case to the preceding problem) be equal to $\mathbb{E} - \frac{1-\varrho A'}{2(1+\varrho)} \times \frac{\alpha}{a}(1+\varrho)^{-n} - \frac{1-\varrho B'}{2(1+\varrho)} \cdot \frac{\beta}{b}(1+\varrho)^{-n} - A^{\circ}B^{\circ} . \frac{\alpha\beta}{ab}(1+\varrho)^{-n} + \frac{1+\overset{\prime}{A} B'}{2(1+\varrho)} \cdot \frac{\overset{\prime}{\alpha}\beta}{ab}(1+\varrho)^{-n} + \frac{1+A^{\circ}\overset{\prime}{B^{\circ}}}{2(1+\varrho)} \cdot \frac{\alpha\overset{\prime}{\beta}}{ab}(1+\varrho)^{-n}$. But, after the decease of C, the expectations arising from these contingencies may be more correctly expressed for all the subsequent years by means of the Lemma above mentioned: for, the chance of receiving the sum at the end of any one of those years will depend on the happening of either of three events: 1° that A and B both fail in the year: 2° that A fails in the year, B having died after C in either of the preceding years: 3° that B fails in the year, A having died after C in either of the preceding years. The probabilities of the happening of these three events in the $(n+1)^{st}$ year, are respectively $\frac{(\alpha-\overset{\prime}{\alpha}).(\beta-\overset{\prime}{\beta})}{ab}$, $\frac{\alpha-\overset{\prime}{\alpha}}{a}\left(1-\phi-\frac{\beta}{b}\right)$* and $\frac{\beta-\overset{\prime}{\beta}}{b}\left(1-\varpi-\frac{\alpha}{a}\right)$:* which being added together and multiplied by $(1+\varrho)^{-(n+1)}$ will give the expectation of receiving the sum at the end of the $(n+1)^{\text{st}}$ year.

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* See the Scholium in page 121.

In like manner, the probabilities of the happening of these events in the $(n+2)^{\text{nd}}$ year are respectively $\frac{(\overset{\prime}{\alpha}-\overset{\prime\prime}{\alpha}).(\overset{\prime}{\beta}-\overset{\prime\prime}{\beta})}{ab}$, $\frac{\overset{\prime}{\alpha}-\overset{\prime\prime}{\alpha}}{a}\left(1-\phi-\frac{\overset{\prime}{\beta}}{b}\right)$ and $\frac{\overset{\prime}{\beta}-\overset{\prime\prime}{\beta}}{b}\left(1-\varpi-\frac{\overset{\prime}{\alpha}}{a}\right)$; which being added together and multiplied by $(1+\varrho)^{-(n+2)}$ will give the expectation of receiving the sum at the end of the $(n+2)^{\text{nd}}$ year. By the same method of reasoning we may find the expectation of receiving the sum at the end of the $(n+3)^{\text{rd}}$ year: and so on to the utmost extent of human life; the sum of all which annual expectations, when reduced to their lowest terms, will be found equal to s multiplied into $(1-\phi)\times\frac{1-\varrho A'}{(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}+(1-\varpi)\times\frac{1-\varrho B'}{(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n}-\frac{1+A'B'}{(1+\varrho)}\cdot\frac{\alpha\beta}{ab}(1+\varrho)^{-n}+A^{\circ}B^{\circ}\cdot\frac{\alpha\beta}{ab}(1+\varrho)^{-n}$: which, being added to the sum of the first n terms of the several collateral series above mentioned, will express the total present value of the given sum in this case required. Whence, such present value will be equal to $s\left[\mathbb{E}+y+z\right]$.

COROLLARY.

§ 294. When all the lives are equal, or of the same age A, the last three quantities in the formula denoted by $\mathbb{E}$ vanish entirely; and the general expression for this case will become $s \times \frac{1-\varrho(A-AA+AAA)}{(1+\varrho)}$.

SCHOLIUM.

§ 295. Since it appears that the present value of the given sum, payable on the contingency mentioned in this problem, added to the present value of the same sum, payable on the contingency mentioned in the preceding problem, are together equal to $s \times \frac{2-\varrho(A+B)}{(1+\varrho)}$: it follows that, the value of one of them being determined, the other may be easily obtained by subtracting such value from the general expression here given: provided the ages of the lives are the same in both instances.

PROBLEM XLI.*

§ 296. To determine the present value of a given sum payable on the decease of A, provided he be the *second* that dies of three given lives A, B, C, and provided C dies before B.†

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* Simpson's *Sup.* Prob. 38, and Prob. 4, in page 74. Dodson, vol. iii. Ques. 42 to 47. Morgan, Prob. 18; and in Phil. Trans. for 1789, page 41.

† That is, provided C dies first, and A second of the three lives; or, provided C be then dead and B living: but I have thought it best, for the sake of uniformity, to preserve the mode of expression adopted in the text.

SOLUTION.

The chance of receiving the given sum at the end of the first year will depend on the happening of either of two events: 1° that all the lives fail in that year, C having died first and A next: 2° that A and C both fail in that year A having died last, and that B lives to the end of it. The probabilities of the happening of these two events are respectively $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{6abc}$, and $\frac{(a-\overset{\prime}{a}).(c-\overset{\prime}{c})\overset{\prime}{b}}{2abc}$; which, being added together and multiplied by $s(1+\rho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But the chance of receiving the given sum at the end of the second and following years will depend on the happening of either of four different events: 1° that all the lives fail in the year, C having died first and A next: 2° that A and C both fail in the year, A having died last, and that B lives to the end of it: 3° that both A and B fail in the year, A having died first, and that C dies in either of the preceding years: 4° that only A dies in the year, B living to the end of it, and C having died in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{6abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{b}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{2ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$: which being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year, to the utmost extent of human life; the sum of all which yearly values will be the total present value of the given sum to be received on the above contingency.

These several yearly expectations being reduced to their least terms, and arranged under each other, will form twelve collateral series: the sum of all which, or the present value required, will be found equal to $s\left[.A^{B} - .A^{BC}\right]$.[*]

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[*] This problem is one of frequent occurrence; and is the first of that series of problems, inserted by Mr. Morgan in the *Philosophical Transactions*, for determining the value of contingent reversions where three lives are involved in the survivorship. Mr. Morgan has, in his usual manner, given *two* formulæ for the solution of the problem; leading us thereby to imagine that the value of the given sum depends on the *seniority* of the lives concerned. This however is not the fact: for, the theorem which I have given in the text is a *general* one; and will apply to either case.

As it may be satisfactory to the reader to see the different results which are obtained from the present investigation and from that pursued by Mr. Morgan, I shall here subjoin his formulæ. But since the same lives are not involved in the same contingencies, I shall (in order to prevent any misunderstanding on this subject) prefix his own statement of the problem.

"To determine the value of a given sum payable on the contingency of C's surviving B, provided the life of A shall be then extinct.

"When either B or C is the oldest of the three lives, the

COROLLARY.

§ 297. When the lives are all equal, or of the same age A, this expression (agreeably to what has been said in the corollary to Prob. XXX) will become equal to $s \times \frac{1-\varrho(3AA-2AAA)}{6(1+\varrho)}$: that is, equal to one

value of the given sum will be $=$ S into $\frac{\varkappa}{6c} \times \left[\frac{\beta(\mathrm{FK}-\mathrm{AFK})}{b} - (\mathrm{BK}-\mathrm{ABK})\right] + \frac{\beta}{3b} \times (\mathrm{FC}-\mathrm{AFC}) - \frac{r-1}{3r} \times (\mathrm{BC}-\mathrm{ABC}) - \frac{m}{3br} \times (\mathrm{PC}-\mathrm{APC}) + \frac{d}{6cr} \times \left[(\mathrm{BT}-\mathrm{ABT}) - \frac{m(\mathrm{PT}-\mathrm{APT})}{b}\right]$.

"When A is the oldest of the three lives, the value will be $=$ S into $\Sigma - \frac{\varkappa}{3c} \times \left[\frac{\alpha(\mathrm{HK}-\mathrm{HBK})}{a} + \frac{\mathrm{AK}-\mathrm{ABK}}{2}\right] - \frac{\alpha}{6a} \times (\mathrm{HC}-\mathrm{HBC}) - \frac{r-1}{3r} \times (\mathrm{AC}-\mathrm{ABC}) + \frac{s}{6ar} \times (\mathrm{NC}-\mathrm{NBC}) + \frac{d}{3cr} \times \left[\frac{\mathrm{AT}-\mathrm{ABT}}{2} + \frac{s(\mathrm{NT}-\mathrm{NBT})}{a}\right]$:" where Σ denotes the same as B^{c} in the notation used in the present treatise, and the other characters the same as already explained in the notes in page 215 and 221.

These formulæ are taken from the last edition of Dr. Price's *Obs. on Rev. Pay.* vol. 1, page 392: the gross and careless errors therein being first corrected. It is hardly necessary to remark (after the repeated observations that I have made of a similar kind) that *all* those expressions in the second formula, and some of those in the first, which involve *two* joint lives are totally useless: since they might be made to vanish, in the manner pointed out in page 216. They consequently now only tend to make the arithmetical solution unnecessarily tedious and confused: and the reader perhaps may be surprised to learn that these two formulæ are precisely the same, disguised under different symbols!!'

sixth of the present value of the given sum payable on the extinction of any two out of the three given lives.

PROBLEM XLII.*

§ 298. To determine the present value of a given sum payable on the decease of A; provided he be the *last* that dies of three given lives A, B, C, and provided C dies before B.†

SOLUTION.

The given sum cannot be received at the end of the first year unless all the three lives fail in that year, and in the order above mentioned; the probability of which is $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{6abc}$; which being multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, the given sum may be received on the happening of either of three different events: 1° that all the lives fail in the year, in the order above mentioned: 2° that A and B both fail in the year, A having died last, and C having failed in either of the preceding years: 3° that only

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* Morgan, Prob. 29; and in Phil. Trans. for 1794, Prob. 6, page 253.

† That is, provided C dies first, B next, and A last.

A dies in the year, and *that both B and C fail in either of the preceding years, C having died first.* The probabilities of the happening of these several events in the second year, are respectively $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{6abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{2ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and $\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{2a}\left(1-\frac{\overset{\prime}{b}}{b}\right)\cdot\left(1-\frac{\overset{\prime}{c}}{c}\right)$: which, being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year during the continuance of the oldest of the given lives: and, if these several annual expectations be reduced to their lowest terms and arranged under each other, they will form fourteen collateral series; the first n terms of which will vary according to the seniority of the lives concerned.

§ 299. *Case* 1. Let A be the oldest of the three lives. In this case the several series here alluded to must be continued to the utmost extent of human life; because the life A is involved in each collateral series. Therefore the value of such series will be equal to s multiplied into $\frac{1-\varrho A}{2(1+\varrho)}+\frac{AB}{2}-\underset{,}{A}B.\frac{\underset{,}{a}}{2a}-\frac{1+AC}{2(1+\varrho)}+\frac{1+\overset{\prime}{A}C}{2(1+\varrho)}.\frac{\overset{\prime}{a}}{a}+\frac{1-\varrho ABC}{6(1+\varrho)}-\frac{1+\overset{\prime}{A}BC}{6(1+\varrho)}.\frac{\overset{\prime}{a}}{a}+\underset{,}{A}BC.\frac{\underset{,}{a}}{6a}+\frac{1+A\overset{\prime}{B}C}{3(1+\varrho)}.\frac{\overset{\prime}{b}}{b}+A\underset{,}{B}C.\frac{\underset{,}{b}}{6b}-\frac{1+AB\overset{\prime}{C}}{6(1+\varrho)}.\frac{\overset{\prime}{c}}{c}-AB\underset{,}{C}.\frac{\underset{,}{c}}{3c}$: an expression which, independent of the common multiple

s, may be reduced to $\frac{1-\varrho(3A+ABC)-3AC}{6(1+\varrho)}+\frac{AB}{2}+\frac{1}{6a}\left[\frac{(2+3\acute{A}C-\acute{A}BC)\acute{a}}{(1+\varrho)}-(3\underset{\prime}{A}B-\underset{\prime}{A}BC)\underset{\prime}{a}\right]+\frac{1}{6b}\times\left[\frac{(1+A\acute{B}C)2\acute{b}}{(1+\varrho)}+A\underset{\prime}{B}C.\underset{\prime}{b}\right]-\frac{1}{6c}\left[\frac{(1+AB\acute{C})\acute{c}}{(1+\varrho)}+A\underset{\prime}{B}C.2\underset{\prime}{c}\right]$; and which I shall denote by $\mathbb{F}$. Therefore when A is the oldest of the three lives, the present value of the given sum will be represented by $s\times\mathbb{F}$.

§ 300. *Case* 2. Let B be the oldest of the three lives. In this case the sum of the first n terms of the several collateral series above mentioned will be found equal to $\mathbb{F}-\frac{1-\varrho A^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}+\frac{1+AC'}{2(1+\varrho)}\times\frac{\alpha\gamma}{ac}(1+\varrho)^{-n}-\frac{1+\acute{A}^{\circ}C'}{2(1+\varrho)}\cdot\frac{\acute{\alpha}\gamma}{ac}(1+\varrho)^{-n}$. But, after the extinction of the life of B, the sum of the expectations for all the subsequent years may be more correctly expressed as in the second case of Prob. XL, by $s.(1-\underset{\cdot}{\phi})\times\frac{1-\varrho A^{\circ}}{(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}$. Therefore if this value be added to the sum of the first n terms of the several series just found, it will make the total present value of the given sum, when B is the oldest of the three lives, equal to $s\left[\mathbb{F}+x\right]$.

§ 301. *Case* 3. Let C be the oldest of the three lives. In this case the sum of the first n terms of the several collateral series above alluded to will be found equal to $\mathbb{F}-\frac{1-\varrho A'}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}-A'B'\cdot\frac{\alpha\beta}{2ab}(1+\varrho)^{-n}+\underset{,}{A}'B'\cdot\frac{\alpha\underset{,}{\beta}}{2ab}(1+\varrho)^{-n}$. But, after the extinction of the life of C, the chance of receiving the given sum at the end of any one year will depend on the happening of two events only: 1° that A and B both fail in the year, A having died last: 2° that only A fails in the year, B having died after C in either of the preceding years: the probabilities of the happening of these events in the $(n+1)^{st}$ year are respectively $\frac{(\alpha-\overset{\prime}{\alpha}).(\beta-\overset{\prime}{\beta})}{2ab}$ and $\frac{\alpha-\overset{\prime}{\alpha}}{a}\left(1-\phi-\frac{\beta}{b}\right)$, which being multiplied by $(1+\varrho)^{-n+1}$ will give the expectation of receiving the sum at the end of the $(n+1)^{st}$ year. In like manner we might proceed to find the expectations of receiving the given sum for very subsequent year to the utmost extent of human life: and the sum of all those expectations will be found equal to s multiplied into $(1-\phi)\times\frac{1-\varrho A'}{(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}-\frac{1+A'B'}{2(1+\varrho)}\times\frac{\alpha\beta}{ab}(1+\varrho)^{-n}+A'B'\cdot\frac{\alpha\beta}{2ab}(1+\varrho)^{-n}+\frac{1+\overset{\prime}{A}'B'}{2(1+\varrho)}\cdot\frac{\overset{\prime}{\alpha}\beta}{ab}\times(1+\varrho)^{-n}-\underset{,}{A}'B'\cdot\frac{\alpha\underset{,}{\beta}}{2ab}(1+\varrho)^{-n}$. Consequently this value, added to the sum of the first n terms of the several collateral series above mentioned, will express the total present value of the given sum, in this case required. Whence, such present value will be equal to $s\Big[\mathbb{F}+z\Big]$.*

COROLLARY.

§ 302. When the three lives are equal, or all of the same age A, the last three quantities in the formulæ denoted by $\mathbb{F}$ vanish altogether, and the general expression in this case will become $s \times \frac{1-\varrho(3A-3AA+AAA)}{6(1+\varrho)}$: that is, equal to one sixth of the present value of the given sum, payable on the extinction of the longest of the three lives.

PROBLEM XLIII.†

§ 303. To determine the present value of a given sum payable on the decease of A, provided he be the *first* or *second* that fails of three given lives A, B, C; and provided C, in the latter case, dies before B.

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[*] Mr. Morgan has divided this problem (I know not for what reason) into *four* distinct cases: and in his investigations of these cases he has fallen into the same errors and absurdities that I have so frequently noticed in the preceding pages. But as it is not my wish to swell the present work with those disgusting repetitions; and as the remarks, which I have already made, apply with nearly equal propriety to *all* Mr. Morgan's problems, I shall not here enter more into detail. It may however be useful perhaps for Mr. Morgan to know that it is *impossible* that L (or the value of an annuity on the longest of the three lives) can *properly* arise in *any* of his problems, unless those lives are *equal*.

[†] Phil. Trans. for 1800, Prob. 4, page 32.

SOLUTION.

The payment of the sum at the end of the first year will depend on the happening of either of four different events: 1° that all the lives fail in the year, A having died first, or C having died first and A second: 2° that A and B fail in the year, A having died first, and that C lives to the end of it: 3° that A and C die in the year, and that B lives to the end of it: 4° that only A dies in the year, and that B and C both live to the end of it. The probabilities of the happening of these several events are respectively $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{2abc}$, $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b})\overset{\prime}{c}}{2abc}$, $\frac{(a-\overset{\prime}{a}).(c-\overset{\prime}{c})\overset{\prime}{b}}{abc}$, and $\frac{(a-\overset{\prime}{a})\overset{\prime}{b}\overset{\prime}{c}}{abc}$: which, being added together and multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, the given sum may be received on the happening of either of six different events; viz. in addition to the four already enumerated, 5° that A and B fail in the year, A having died first, and that C dies in either of the preceding years: 6° that only A dies in the year, B living to the end of it, and C having died in either of the preceding years. The probabilities of the happening of these six different events in the second year are respectively $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{c}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{b}}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{2ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and

$\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$: which being added together, and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. And so for every subsequent year to the utmost extent of human life: the sum of all which values will be the total present value of the sum required: and which will be found equal to $s \times \mathscr{A}^{B}$, or to the present value of the given sum payable on the decease of A provided he be the first that dies of the two lives A, B; as found by Prob. XXVII. The truth of which is evident: for, the payment of the given sum can be prevented only by the event of B dying before A.*

PROBLEM XLIV.†

§ 304. To determine the present value of a given sum payable on the decease of A, provided he be the *second* or *third* that fails of three given lives A, B, C; and provided C dies before B.

SOLUTION.

The given sum may be received at the end of the first year, on the happening of either of two events: 1° that all the lives become extinct in that year, C having died first: 2° that A and C both fail in that year, C having died first, and that B lives to the end of it. The probabilities of the happening of these events are respectively $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{3abc}$, and $\frac{(a-\overset{\prime}{a}).(c-\overset{\prime}{c})\overset{\prime}{b}}{2abc}$: which, being added together and multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

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* It may be here useful to remark that the present value found by this problem is equal to the sum of the two values found by Prob. XXIX and XLI: and their agreement in this particular confirms the accuracy of the investigation.

† Phil. Trans. for 1800, Prob. 5, page 33.

But, in the second and following years the given sum may be received on the happening of either of five different events: 1° that all the lives fail in the year, C having died first: 2° that A and C both fail in the year, C having died first, and that B lives to the end of it: 3° that A and B both die in the year, C having died in either of the preceding years: 4° that only A dies in the year, B living to the end of it, and C having died in either of the preceding years: 5° that only A fails in the year, and *that B and C both fail in either of the preceding years, C having died first.* The probabilities of the happening of these several events in the second year are respectively $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{3abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{b}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{ab}\cdot\left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})}{2a}\left(1-\frac{\overset{\prime}{b}}{b}\right)\cdot\left(1-\frac{\overset{\prime}{c}}{c}\right)$: which, being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every succeeding year: and if these several annual expectations be reduced to their lowest terms and arranged under each other, they will form fourteen collateral series; the sum of the first n terms of which will vary according to the seniority of the lives concerned.

§ 305. *Case* 1. Let A be the oldest of the three lives. In this case, the terms of the several series here alluded to must be continued to the utmost extent of human life, because the life of A is involved in each collateral series: therefore the sum of it will be equal to s multiplied into $\frac{1-\varrho A}{2(1+\varrho)} + \frac{1+AB}{2(1+\varrho)} - \frac{1+\acute{A}B}{2(1+\varrho)}\cdot\frac{\acute{a}}{a} - \frac{1+AC}{2(1+\varrho)} + \frac{1+\acute{A}C}{2(1+\varrho)}\cdot\frac{\acute{a}}{a} - \frac{1-\varrho ABC}{6(1+\varrho)} + \frac{1+\dot{A}BC}{6(1+\varrho)}\cdot\frac{\acute{a}}{a} - \underset{\prime}{A}BC.\frac{\underset{\prime}{a}}{6a} + \frac{1+\ \dot{B}C}{6(1+\varrho)}\cdot\frac{\acute{b}}{b} + \underset{\prime}{A}BC.\frac{\underset{\prime}{b}}{3b} - \frac{1+AB\dot{C}}{3(1+\varrho)}\cdot\frac{\acute{c}}{c} - \underset{\prime}{A}BC.\frac{\underset{\prime}{c}}{6c}$: an expression which, independent of the common multiple s, may be reduced, as in the former cases, to $\frac{2-\varrho(3A-ABC)+3AB-3AC}{6(1+\varrho)} + \frac{1}{6a}\times \left[\frac{(1-3\acute{A}B+3\acute{A}C+\acute{A}BC)\acute{a}}{(1+\varrho)} - \underset{\prime}{A}BC.\underset{\prime}{a}\right] + \frac{1}{6b}\left[\frac{(1+A\acute{B}C)\acute{b}}{(1+\varrho)} + \underset{\prime}{A}BC.2\underset{\prime}{b}\right] - \frac{1}{6c}\left[\frac{(1+AB\acute{C})2\acute{c}}{(1+\varrho)} + AB\underset{\prime}{C}.\underset{\prime}{c}\right]$; and which I shall denote by 𝔊. Therefore, when A is the oldest life, the required value will be represented by $s \times$ 𝔊.

§ 306. *Case* 2. Let B be the oldest of the three lives. In this case the first n terms of the several collateral series above mentioned will be equal to $\mathfrak{G} - \frac{1-\varrho A^{\prime}}{2(1+\varrho)} \cdot \frac{\alpha}{a}(1+\varrho)^{-n} + \frac{1+A^{\circ}C^{\circ}}{2(1+\varrho)} \cdot \frac{\alpha\gamma}{ac}(1+\varrho)^{-n} - \frac{1+\overset{\prime}{A}{}^{\circ}C^{\circ}}{2(1+\varrho)} \cdot \frac{\overset{\prime}{\alpha}\gamma}{ac}(1+\varrho)^{-n}$. But after the decease of B the expectations for all the subsequent years may be more correctly expressed by means of the first Lemma in the fifth Chapter: for, since the chance of receiving the sum at the end of any one of those years depends on A's failing in the year, C having died before B in either of the preceding years (the probability of which is, by the Lemma denoted by $1-\phi$) we shall find that the sum of the expectations for those years continued to the utmost extent of human life, will be equal to $s.(1-\phi) \times \frac{1-A^{\prime}}{(1+\varrho)} \cdot \frac{\alpha}{a}(1+\varrho)^{-n}$. Consequently, this value added to the sum of the first n terms of the series above found, will express the total present value of the given sum in this case required: and which will be found equal to $s\left[\mathfrak{G} + x\right]$.

§ 307. *Case* 3. Let C be the oldest of the three lives. In this case the first n terms of the several collateral series above mentioned will be equal to $\mathfrak{G} - \frac{1-\varrho A^{\circ}}{2(1+\varrho)} \cdot \frac{\alpha}{a}(1+\varrho)^{-n} - \frac{1+A^{\circ}B^{\prime}}{2(1+\varrho)} \cdot \frac{\alpha\beta}{ab}(1+\varrho)^{-n} + \frac{1+\overset{\prime}{A}{}^{\circ}B^{\prime}}{2(1+\varrho)} \cdot \frac{\overset{\prime}{\alpha}\beta}{ab}(1+\varrho)^{-n}$. But, after the decease of C, the chance of receiving the given sum at the end of any subsequent year will depend solely on the death of A, C having died before B in either of the preceding years; whence, the sum of the expectations for all the subsequent years will, as in the preceding case, be denoted by $s(1-\varphi)\times\frac{1-A'}{(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}$. Consequently, this value, added to the sum of the first n terms of the series above found, will represent the total present value of the given sum in this case required: and which will be found equal to $s\left[\mathfrak{G}+z\right]$.*

COROLLARY.

§ 308. When the three lives are equal, or all of the same age A, the last three quantities in the formula denoted by $\mathfrak{G}$ destroy each other, and the general expression will in this case become $s\times\frac{2-\varrho(3A-AAA)}{6(1+\varrho)}$.

PROBLEM XLV.†

§ 309. To determine the present value of a given sum payable on the decease of A, provided he be the *first* or *last* that fails of three given lives A, B, C; and provided C, in the latter case, dies before B.

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* It may be here useful to remark that the present value found by this problem is equal to the sum of the two values found by Prob. XLI and XLII: and their agreement in this particular is a proof of the accuracy of the investigation.

† Phil. Trans. for 1800, Prob. 6, page 37.

SOLUTION.

The given sum may be received at the end of the first year on the happening of either of four different events: 1° that all the lives fail in that year, A having died first, or C having died first, B second, and A last: 2° that A and B fail in that year, A having died first, and that C lives to the end of it: 3° that A and C fail in that year, A having died first, and that B lives to the end of it: 4° that only A dies in that year, and that B and C both live to the end of it. The probabilities of the happening of these several events are respectively $\frac{(a-\acute{a}).(b-\acute{b}).(c-\acute{c})}{2abc}$, $\frac{(a-\acute{a}).(b-\acute{b})\acute{c}}{2abc}$, $\frac{(a-\acute{a}).(c-\acute{c})\acute{b}}{2abc}$, and $\frac{(a-\acute{a})\acute{b}\acute{c}}{abc}$: which, being added together and multiplied by $s(1+\rho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years the given sum may be received on the happening of either of six different events: 1° that all the lives fail in the year, A having died first, or C having died first, B second, and A last: 2° that A and B fail in the year, A having died first, and that C lives to the end of it: 3° that A and C fail in the year, A having died first, and that B lives to the end of it: 4° that only A dies in the year, and that B and C both live: 5° that A and B both fail in the year, A having died last, and C having failed in either of the preceding years: 6° that only A fails in the year, and *that B and C both fail in either of the preceding years, C having died first.* The probabilities of the happening of these several events in the second year are respectively $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{c}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})\overset{\prime\prime}{b}}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{b}\overset{\prime\prime}{c}}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{2ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and $\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{2a}\left(1-\frac{\overset{\prime}{b}}{b}\right) \times \left(1-\frac{\overset{\prime}{c}}{c}\right)$: which, being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner, we may find the expectation of receiving the sum at the end of the third and every subsequent year, during the continuance of the oldest of the given lives: and, if these several annual expectations be reduced to their lowest terms, and arranged under each other, they will form twelve collateral series; the first n terms of which will vary according to the seniority of the lives concerned.

§ 310. *Case* 1. Let A be the oldest of the three lives. In this case, the several collateral series here mentioned must be continued to the utmost extent of human life, because the life A is involved in every term of it: therefore the sum of it will be equal to s multiplied into $\frac{1-\varrho A}{2(1+\varrho)}+\frac{AB}{2}-{}_{\prime}AB.\frac{\underset{\prime}{a}}{2a}-\frac{1+AC}{2(1+\varrho)}+\frac{1+\overset{\prime}{A}C}{2(1+\varrho)}.\frac{\overset{\prime}{a}}{a}+\frac{1-\varrho ABC}{2(1+\varrho)}-\frac{1+\overset{\prime}{A}BC}{2(1+\varrho)}.\frac{\overset{\prime}{a}}{a}+{}_{\prime}ABC.\frac{\underset{\prime}{a}}{2a}+$

$\frac{1+A\overset{\prime}{B}C}{2(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{b} - AB\underset{\prime}{C}\cdot\frac{\overset{c}{\prime}}{2c}$: an expression which, independent of the common multiple *s*, may be reduced to $\frac{1-\varrho(A+ABC)-AC}{2(1+\varrho)} + \frac{AB}{2} + \frac{1}{2a}\left[\frac{(\overset{\prime}{A}C-\overset{\prime}{A}BC)\overset{\prime}{a}}{(1+\varrho)} - (\underset{\prime}{A}B-\underset{\prime}{A}BC)\underset{\prime}{a}\right] + \frac{1+A\overset{\prime}{B}C}{2(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{b} - AB\underset{\prime}{C}\cdot\frac{\overset{c}{\prime}}{2c}$; and which I shall denote by $\mathbb{H}$. Therefore, when A is the oldest of the three lives, the value required will be equal to $s \times \underset{\prime}{\mathbb{H}}$.*

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* It is really amusing to observe the manner in which Mr. Morgan has summed up the several series arising from this problem: and since this manner is by no means unusual with him, and as it gives me an opportunity of explaining more fully what I have already said, in the preceding pages, on his singular and confused method of investigating such cases, I shall here (for the last time) enter once more on the subject.

He makes the total value of the series, continued to the utmost extent of human life, equal to S into $\frac{r-1}{2r}(\mathrm{V-A-ABC}) + \frac{\mathrm{AC}}{2} - \frac{\mathrm{AB}}{2r} + \frac{md}{2bcr}(1+\mathrm{APT}) - \frac{d}{2cr}(1+\mathrm{AT}+\mathrm{BT}-\mathrm{ABT}) - \frac{\beta\varkappa}{2bc} \times \mathrm{AFK} + \frac{\beta}{2b}(\mathrm{AF}+\mathrm{FC}-\mathrm{AFC})$: See *Phil. Trans.* for 1800, page 37. Now, this is certainly the correct value of the series: but, it is necessary to remark that $\frac{\beta.\mathrm{FC}}{2b}$ is equal to $\frac{d}{2cr}(1+\mathrm{BT})$; and, as these two quantities are used with *contrary signs*, it is evident that they ought not to have been introduced into the equation. Consequently the formula should have been reduced to $\frac{r-1}{2r} \times (\mathrm{V-A-ABC}) + \frac{\mathrm{AC}}{2} - \frac{\mathrm{AB}}{2r} + \frac{md}{2bcr}(1+\mathrm{APT}) - \frac{d}{2cr} \times (\mathrm{AT}-\mathrm{ABT}) - \frac{\beta\varkappa}{2bc} \times \mathrm{AFK} + \frac{\beta}{2b}(\mathrm{AF}-\mathrm{AFC})$.

The reader will observe that this is the total value of the series,

§ 311. *Case* 2. Let B be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned will be equal to $\mathbb{H} - \frac{1-\varrho A'}{2(1+\varrho)} \cdot \frac{\alpha}{a}(1+\varrho)^{-n} + \frac{1+A'C'}{2(1+\varrho)} \times \frac{\alpha\gamma}{ac}(1+\varrho)^{-n} - \frac{1+\acute{A}'C''}{2(1+\varrho)} \cdot \frac{\acute{\alpha}\gamma}{ac}(1+\varrho)^{-n}$. But, after the decease of B, the value of the expectations for all the sub-

continued to the utmost extent of human life; and therefore that it will truly denote the value of the given sum *when* A *is the oldest of the three lives.* Mr. Morgan, however, when he comes to consider that case, says that the symbols *must be changed;* and he then makes the given sum equal to S into $\frac{r-1}{2r}(\mathrm{V}-\mathrm{A}-\mathrm{ABC}) + \frac{\mathrm{AC}}{2} - \frac{\mathrm{AB}}{2r} - \frac{\varkappa.\mathrm{AK}}{2c} + \frac{a.\mathrm{HBC}}{2a} - \frac{d}{2cr}(\mathrm{AT}-\mathrm{ABT}) + \frac{s}{2ar}(1+\mathrm{NC}+\mathrm{NB}-\mathrm{NBC}) - \frac{ds}{2acr}(1+\mathrm{NBT})$. Now, a moment's attention to this formula will show us that it does not in reality differ from the preceding one: for

$$\frac{md}{2bcr}(1+\mathrm{APT}) = \frac{a.\mathrm{HBC}}{2a}$$

$$\frac{\beta}{2b}\left(\mathrm{AF} - \frac{\varkappa}{c}\,\mathrm{AFK}\right) = \frac{s}{2ar}(\mathrm{NB}-\mathrm{NBC})$$

$$\frac{\beta.\mathrm{AFC}}{2b} = \frac{ds}{2acr}(1+\mathrm{NBT});$$

whence, these values may be safely substituted for each other, since they by no means denote *different* quantities, but arise merely from the different methods of summing up the *same* series, agreeably to what I have before remarked. But, as the first formula had been already deduced by him, and was stated at length, it certainly did not become necessary to confuse the subject by introducing a second.

The most remarkable circumstance, however, attending this sequent years may be more correctly expressed, as in the second case to Prob. XL, by $s(1-\phi)\frac{1-\varrho A^{\prime}}{(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}$: and which being added to the sum of the first n terms found as above, will make the total present value of the given sum, when B is the oldest of the three lives, equal to $s\left[\mathbb{H}+x\right]$.

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investigation, is that (after discarding the two useless quantities $\frac{\beta.\mathrm{FC}}{2b}-\frac{d}{2ar}(1+\mathrm{BT})$ from this second formula) he should insert in their stead two *other* useless quantities, $\frac{\varkappa.\mathrm{AK}}{2c}$ and $\frac{s}{2ar}(1+\mathrm{NC})$: which, being equal to each other, and used with *contrary signs*, ought of course to have been discarded also. Surely Mr. Morgan could not have been writing either for the *convenience* or the *information* of mankind!!!

I shall here take the opportunity of repeating the observation made in the note in page 219; that, although the quantities here alluded to are strictly and mathematically *equal*, and being used with contrary signs *destroy each other*, yet (owing to the imperfection of the Tables which show the value of Life Annuities) they oftentimes produce a *real* expression when solved *arithmetically*. Thus $\frac{s}{ar}(1+\mathrm{NC})-\frac{\varkappa.\mathrm{AK}}{c}$ (which, expressed by the characters made use of in this work, becomes $\frac{1+\acute{A}\overset{\prime}{C}}{(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a}-\underset{\prime}{A}C.\frac{\overset{c}{\prime}}{c}$) is, from what has been said in the note in page 186, equal to 0: but if reduced to *numbers* (taking the age of A 15 years, the age of C 18 years, the rate of interest 4 per cent, and the probabilities of living as at *Northampton*) it will become equal to 13·362 − 13·348 = 014. This fact will sufficiently show how necessary it is to discard all *superfluous* quantities: and that Mr. Morgan's formulæ are, at best, but approximations.

§ 312. *Case* 3. Let C be the oldest of the three lives. In this case, the first n terms of the several collateral series above mentioned will be equal to $\mathbb{H} - \frac{1-\varrho A'}{2(1+\varrho)} \cdot \frac{\alpha}{a}(1+\varrho)^{-n} - A'B' \cdot \frac{\alpha\beta}{2ab}(1+\varrho)^{-n} + {}_{\prime}A'B' \cdot \frac{\overset{\alpha\beta}{\prime}}{2ab}(1+\varrho)^{-n}$. But, after the decease of C the chance of receiving the given sum at the end of any subsequent year will depend on the happening of either of two events: 1° that A dies after B in the year: 2° that only A dies in the year, B having died after C in either of the preceding years. The sum of the expectations on the happening of these events has been already found, by the third case in Prob. XLII, to be equal to $(1-\phi) \times \frac{1-\varrho A'}{(1+\varrho)} \cdot \frac{\alpha}{a}(1+\varrho)^{-n} - \frac{1+A^{\circ}B^{\circ}}{2(1+\varrho)} \times \frac{\alpha\beta}{ab}(1+\varrho)^{-n} + A^{\circ}B^{\circ} \cdot \frac{\alpha\beta}{2ab}(1+\varrho)^{-n} + \frac{1+\acute{A}'B'}{2(1+\varrho)} \cdot \frac{\acute{\alpha}\beta}{ab} \times (1+\varrho)^{-n} - {}_{\prime}A^{\circ}B^{\circ} \cdot \frac{\overset{\alpha\beta}{\prime}}{2ab}(1+\varrho)^{-n}$: which, being added to the sum of the first n terms of the several collateral series above mentioned, will make the total present value required equal to $s\left[\mathbb{H} + z\right]$.*

COROLLARY.

§ 313. When the three lives are equal, or all of the same age A, the last three quantities in the formula denoted by II destroy each other, and the general expression in this case will become equal to $s \times \frac{1-\varrho(A-AA+AAA)}{2(1+\varrho)}$.

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* It may be useful to remark that the present value found by this problem is equal to the sum of the two values found by Prob. XXIX and XLII; which confirms the accuracy of the investigation.

PROBLEM XLVI.*

§ 314. To determine the present value of a given sum payable on the decease of A *and* B, provided they be the *first* that fail of three given lives A, B, C.

SOLUTION.

In order to receive the sum at the end of the first year, either of these two events must happen: 1° that all the lives fail in that year, C having died last: 2° that the two lives A and B fail in that year, and that C lives to the end of it. The probabilities of the happening of these two events are respectively $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{3abc}$, and $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b})\overset{\prime}{c}}{abc}$: which, being added together and multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of six different events: 1° that all the lives fail in the year,

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* Simpson's *Sup*. Prob. 39. Dodson, vol. iii. Ques. 48. Morgan, Prob. 22; and in Phil. Trans. for 1791, Prob. 4, page 258.

C having died last: 2° that A and B both fail in the year, and that C lives to the end of it: 3° that A and C both fail in the year, A having died first, and that B dies in either of the preceding years: 4° that B and C both fail in the year, B having died first, and that A dies in either of the preceding years: 5° that only A dies in the year, C living to the end of it, and B having died in either of the preceding years: 6° that only B dies in the year, C living to the end of it, and A having died in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively $\frac{(a'-a'').(b'-b'').(c'-c'')}{3abc}$, $\frac{(a'-a'').(b'-b'')c''}{abc}$, $\frac{(a'-a'').(c'-c'')}{2ac}\left(1-\frac{b'}{b}\right)$, $\frac{(b'-b'').(c'-c'')}{2bc}\times\left(1-\frac{a'}{a}\right)$, $\frac{(a'-a'')c''}{ac}\left(1-\frac{b'}{b}\right)$, and $\frac{(b'-b'')c''}{bc}\left(1-\frac{a'}{a}\right)$: which being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year, to the utmost extent of human life: and if these several expectations be reduced to their lowest terms and arranged under each other they will form sixteen collateral series, the sum of which will be found equal to $s\left[\mathcal{A}^C+\mathcal{B}^C-\mathcal{AB}^C\right]$.

COROLLARY.

§ 315. When the lives are all equal to each other, or of the same age A, this expression (agreeably to what has been said in the corollary to Prob. XXX) will become equal to $s \times \frac{1-\varrho(3AA-2AAA)}{3(1+\varrho)}$: that is, equal to one third of the present value of the given sum payable on the extinction of any two out of the three given lives.

PROBLEM XLVII.*

§ 316. To determine the present value of a given sum payable on the decease of A *and* B, provided they be the *last* that fail of three given lives A, B, C.

SOLUTION.

It is evident that the given sum cannot be received at the end of the first year unless all the lives become extinct, C having died first: the probability of which is $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{3abc}$: and which, being multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of four different events: 1° that all the lives fail in the year, C having died first: 2° that A and B fail in the year, C having died in either of the preceding years:

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* Morgan, Prob. 23; and in Phil. Trans. for 1794, Prob. 5, page 247.

3° that only A fails in the year, and *that B and C both fail in either of the preceding years, B having died last:* 4° that only B fails in the year, and *that A and C both fail in either of the preceding years, A having died last.* The probabilities of the happening of these several events, in the second year, are respectively $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{3abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{2a}\left(1-\frac{\overset{\prime}{b}}{b}\right)\cdot\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and $\frac{\overset{\prime}{b}-\overset{\prime\prime}{b}}{2b}\left(1-\frac{\overset{\prime}{a}}{a}\right)\cdot\left(1-\frac{\overset{\prime}{c}}{c}\right)$: which, being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner, we may find the expectation of receiving the sum at the end of the third and every subsequent year: and, if these several expectations be reduced to their lowest terms and arranged under each other, they will form nineteen collateral series; the sum of which, if continued to the utmost extent of human life, would be equal to s multiplied into $\frac{1-\varrho A}{2(1+\varrho)}+\frac{1-\varrho B}{2(1+\varrho)}+AB-\frac{1+\overset{\prime}{A}B}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a}-\frac{1+A\overset{\prime}{B}}{2(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{b}-\frac{1+AC}{2(1+\varrho)}+\frac{1+\overset{\prime}{A}C}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a}-\frac{1+BC}{2(1+\varrho)}+\frac{1+\overset{\prime}{B}C}{2(1+\varrho)}\times\frac{\overset{\prime}{b}}{b}+\frac{1-\varrho ABC}{3(1+\varrho)}+\frac{1+\overset{\prime}{A}BC}{6(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a}+\underset{\prime}{A}BC.\frac{\underset{\prime}{a}}{3a}+\frac{1+A\overset{\prime}{B}C}{6(1+\varrho)}\times\frac{\overset{\prime}{b}}{b}+A\underset{\prime}{B}C.\frac{\underset{\prime}{b}}{3b}-\frac{1+AB\overset{\prime}{C}}{3(1+\varrho)}\cdot\frac{\overset{\prime}{c}}{c}-AB\underset{\prime}{C}.\frac{\underset{\prime}{2c}}{3c}$: an expression which, independent of the common multiple s, may be reduced to $\frac{2-\varrho(3A+3B+2ABC)-3AC-3BC}{6(1+\varrho)}$

$+ AB + \frac{1}{6a}\left[\frac{(1-3\overset{\prime}{A}B+3\overset{\prime}{A}C+\overset{\prime}{A}BC)\overset{\prime}{a}}{(1+\varrho)} + \underset{\prime}{A}BC.2\underset{\prime}{a}\right]$ $+ \frac{1}{6b}\left[\frac{(1-3A\overset{\prime}{B}+3\overset{\prime}{B}C+A\overset{\prime}{B}C)\overset{\prime}{b}}{(1+\varrho)} + A\underset{\prime}{B}C.2\underset{\prime}{b}\right] - \frac{1}{3c} \times$ $\left[\frac{(1+AB\overset{\prime}{C})\overset{\prime}{c}}{(1+\varrho)} + AB\underset{\prime}{C}.2\underset{\prime}{c}\right]$; and which I shall denote by $\mathbb{I}$.

§ 317. *Case* 1. Let A be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned will be equal to $\mathbb{I} - \frac{1-\varrho B'}{2(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n} + \frac{1+B'C'}{2(1+\varrho)} \times$ $\frac{\beta\gamma}{bc}(1+\varrho)^{-n} - \frac{1+\overset{\prime}{B}'C'}{2(1+\varrho)}\cdot\frac{\overset{\prime}{\beta}\gamma}{bc}(1+\varrho)^{-n}$. But, after the decease of A, the sum of the expectations for all the subsequent years will, as in the first case of Prob. XL, be more correctly expressed by $s(1-\varpi) \times \frac{1-\varrho B'}{(1+\varrho)} \times$ $\frac{\beta}{b}(1+\varrho)^{-n}$: and which, being added to the first n terms just found, will make the total present value, when A is the oldest life, equal to $s\left[\mathbb{I} + {}'w\right]$.

§ 318. *Case* 2. Let B be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned will be equal to $\mathbb{I} - \frac{1-\varrho A'}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n} + \frac{1+A'C'}{2(1+\varrho)} \times$

$\frac{\alpha\gamma}{ac}(1+\varrho)^{-n} - \frac{1+\acute{A}^{\circ}C^{\circ}}{2(1+\varrho)}\cdot\frac{\acute{\alpha}\gamma}{ac}(1+\varrho)^{-n}$. But, after the decease of B, the sum of the expectations for all the subsequent years will, as in the second case of Prob. XL, be more correctly expressed by $s(1-\phi) \times \frac{1-\varrho A^{\circ}}{(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}$: and which, being added to the first n terms just found, will make the total present value, when B is the oldest life, equal to $s\left[\mathbb{I} + x\right]$.

§ 319. *Case* 3. Let C be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned will be equal to $\mathbb{I} - \frac{1-\varrho A^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n} - \frac{1-\varrho B^{\circ}}{2(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n} - A^{\circ}B^{\circ}.\frac{\alpha\beta}{ab}(1+\varrho)^{-n} + \frac{1+\acute{A}^{\circ}B^{\circ}}{2(1+\varrho)}\cdot\frac{\acute{\alpha}\beta}{ab}(1+\varrho)^{-n} + \frac{1+A^{\circ}\acute{B}^{\circ}}{2(1+\varrho)} \times \frac{\alpha\acute{\beta}}{ab}(1+\varrho)^{-n}$. But, after the decease of C, the sum of the expectations for all the subsequent years will, as in the third case of Prob. XL, be more correctly expressed by s multiplied into $(1-\phi) \times \frac{1-\varrho A^{\circ}}{(1+\varrho)} \times \frac{\alpha}{a}(1+\varrho)^{-n} + (1-\varpi) \times \frac{1-\varrho B^{\circ}}{(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n} - \frac{1+A^{\circ}B^{\circ}}{(1+\varrho)} \times \frac{\alpha\beta}{ab}(1+\varrho)^{-n} + A^{\circ}B^{\circ}\cdot\frac{\alpha\beta}{ab}(1+\varrho)^{-n}$: and which, being added to the first n terms just found, will make the total present value, when C is the oldest life, equal to $s\left[\mathbb{I} + y + z\right]$.

COROLLARY.

§ 320. When all the lives are equal, or of the same age A, the last three quantities in the formula denoted by II destroy each other, and the general expression in this case will become equal to $s \times \frac{1-\varrho(3A-3AA+AAA)}{3(1+\varrho)}$: that is, equal to one third of the present value of the given sum to be received on the extinction of the longest of the three lives.

PROBLEM XLVIII.*

§ 321. To determine the present value of a given sum payable on the decease of A *and* B, provided they be the *first* and *last* that fail of three given lives A, B, C.

SOLUTION.

The given sum cannot be received at the end of the first year, unless all the three lives are extinct, C having died second; the probability of which is $\frac{(a-\overset{\prime}{a}).(b-\overset{\prime}{b}).(c-\overset{\prime}{c})}{3abc}$: and which being multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

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[*] This case appears to have been omitted by Mr. Morgan: and no solution of it, prior to the present one, has hitherto been published.

But, in the second and following years the given sum may be received on the happening of either of five different events: 1° that all the lives fail in the year, C having died second: 2° that A and C both fail in the year, A having died last, and B having failed in either of the preceding years: 3° that B and C both fail in the year, B having died last, and A having failed in either of the preceding years: 4° that only A dies in the year, and *that B and C both fail in either of the preceding years, B having died first:* 5° that only B dies in the year, and *that A and C both fail in either of the preceding years, A having died first.* The probabilities of the happening of these several events in the second year are respectively $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{3abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2bc}\times\left(1-\frac{\overset{\prime}{a}}{a}\right)$, $\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{2a}\left(1-\frac{\overset{\prime}{b}}{b}\right)\cdot\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and $\frac{\overset{\prime}{b}-\overset{\prime\prime}{b}}{2b}\left(1-\frac{\overset{\prime}{a}}{a}\right)\times\left(1-\frac{\overset{\prime}{c}}{c}\right)$: which being added together, and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year: and if these several annual expectations be reduced to their lowest terms, and arranged under each other, they will form nineteen collateral series; the sum of which, if continued to the utmost extent of human life, would be equal to s multiplied into $\frac{1-\varrho A}{2(1+\varrho)}+\frac{1-\varrho B}{2(1+\varrho)}-$

$$\frac{1+AB}{(1+\varrho)} + \frac{1+\overset{\prime}{A}B}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a} + \frac{1+A\overset{\prime}{B}}{2(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{b} + \frac{AC}{2} - \underset{,}{A}C\cdot\frac{\underset{,}{a}}{2a} +$$

$$\frac{BC}{2} - \underset{,}{B}C\cdot\frac{\underset{,}{b}}{2b} + \frac{1-\varrho ABC}{3(1+\varrho)} - \frac{1+\overset{\prime}{A}BC}{3(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a} - \underset{,}{A}BC\cdot\frac{\underset{,}{a}}{6a}$$

$$- \frac{1+A\overset{\prime}{B}C}{3(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{b} - A\underset{,}{B}C\cdot\frac{\underset{,}{b}}{6b} + \frac{1+AB\overset{\prime}{C}}{3(1+\varrho)}\cdot\frac{2\overset{\prime}{c}}{c} + AB\underset{,}{C}\cdot\frac{\underset{,}{c}}{3c}.$$

But this expression, independent of the common multiple s, may be reduced to $\frac{2-\varrho(3A+3B+2ABC)-6AB}{6(1+\varrho)} +$
$\frac{AC+BC}{2} + \frac{1}{6a}\left[\frac{(1+3\overset{\prime}{A}B-2\overset{\prime}{A}BC)\overset{\prime}{a}}{(1+\varrho)} - (3\underset{,}{A}C+\underset{,}{A}BC)\underset{,}{a}\right]$
$+ \frac{1}{6b}\left[\frac{(1+3A\overset{\prime}{B}-2A\overset{\prime}{B}C)\overset{\prime}{b}}{(1+\varrho)} - (3\underset{,}{B}C+A\underset{,}{B}C)\underset{,}{b}\right] + \frac{1}{3c}\times$
$\left[\frac{(1+AB\overset{\prime}{C})2\overset{\prime}{c}}{(1+\varrho)} + AB\underset{,}{C}\cdot\underset{,}{c}\right]$; and I shall denote it by $\mathbb{K}$.

§ 322. *Case* 1. Let A be the oldest of the three lives. In this case, the first n terms of the several collateral series, here alluded to, will be equal to $\mathbb{K} - \frac{1-\varrho B^{\circ}}{2(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n} - B'C^{\circ}\cdot\frac{\beta\gamma}{2bc}(1+\varrho)^{-n} + B\underset{,}{C}^{\circ}\cdot\frac{\underset{,}{\beta}\gamma}{2bc}(1+\varrho)^{-n}$. But, after the decease of A, the chance of receiving the sum at the end of any subsequent year will depend on the happening of two different events: 1° that B and C both fail in the year, B having died last: 2° that only B dies in the year, C having died after A in either of the preceding years. The probabilities of the happening of these two events in the $(n+1)^{st}$ year, are respectively $\frac{(\beta-\overset{\prime}{\beta}).(\gamma-\overset{\prime}{\gamma})}{2bc}$, and

$\frac{\beta-\overset{\prime}{\beta}}{b}\left(\varpi-\frac{\gamma}{c}\right)$;* which being added together and multiplied by $s(1+\varrho)^{-(n+1)}$ will give the expectation of receiving the sum at the end of the $(n+1)^{\text{st}}$ year.

In like manner, the probabilities of the happening of these two events in the $(n+2)^{\text{nd}}$ year, are respectively $\frac{(\overset{\prime}{\beta}-\overset{\prime\prime}{\beta}).(\overset{\prime}{\gamma}-\overset{\prime\prime}{\gamma})}{2bc}$, and $\frac{\overset{\prime}{\beta}-\overset{\prime\prime}{\beta}}{b}\left(\varpi-\frac{\overset{\prime}{\gamma}}{c}\right)$; which being added together and multiplied by $s(1+\varrho)^{-(n+2)}$ will give the expectation of receiving the sum at the end of the $(n+2)^{\text{nd}}$ year: and so on for all the subsequent years to the utmost extremity of human life. But, the sum of all these annual expectations will be found to be equal to s multiplied into $\varpi \times \frac{1-\varrho B'}{(1+\varrho)} \times \frac{\beta}{b}(1+\varrho)^{-n} - \frac{1+B'C'}{2(1+\varrho)}\cdot\frac{\beta\gamma}{bc}(1+\varrho)^{-n} + B'C' \times \frac{\beta\gamma}{2bc}(1+\varrho)^{-n} + \frac{1+\overset{\prime}{B}C'}{2(1+\varrho)}\cdot\frac{\overset{\prime}{\beta}\gamma}{bc}(1+\varrho)^{-n} - \underset{\prime}{B}C' \times \frac{\beta\gamma}{2bc}(1+\varrho)^{-n}$: and which, being added to the first n terms of the several collateral series above mentioned, will make the total present value of the given sum, when A is the oldest life, equal to $s\left[\mathbb{K}-x\right]$.

§ 323. *Case* 2. Let B be the oldest of the three lives. In this case, the first n terms of the several collateral series above mentioned will be equal to

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* See the Scholium in page 12[illegible]

$\mathbb{K} - \frac{1-\varrho A^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n} - A^{\circ}C^{\circ}\cdot\frac{\alpha\gamma}{2ac}(1+\varrho)^{-n} + \underset{\prime}{A}^{\circ}C^{\circ}\cdot\frac{\alpha\underset{\prime}{\gamma}}{2ac}(1+\varrho)^{-n}$. But, after the decease of B, the chance of receiving the sum at the end of any subsequent year will depend on the happening of two different events: 1° that A and C both fail in the year, A having died last: 2° that only A dies in the year, C having died after B in either of the preceding years. The probabilities of the happening of these two events in the $(n+1)^{\text{st}}$ year, are respectively $\frac{(\alpha-\overset{\prime}{\alpha}).(\gamma-\overset{\prime}{\gamma})}{2ac}$, and $\frac{\alpha-\overset{\prime}{\alpha}}{a}(\phi-\frac{\gamma}{c})$; which being added together and multiplied by $s(1+\varrho)^{-(n+1)}$ will give the expectation of receiving the sum at the end of the $(n+1)^{\text{st}}$ year.

In like manner, we might find the expectation of receiving the sum at the end of the $(n+2)^{\text{nd}}$ and every subsequent year, to the utmost extremity of human life; the sum of all which annual expectations will be found equal to s multiplied into $\phi \times \frac{1-\varrho A^{\circ}}{(1+\varrho)} \times \frac{\alpha}{a}(1+\varrho)^{-n} - \frac{1+A^{\circ}C^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha\gamma}{ac}(1+\varrho)^{-n} + A^{\circ}C^{\circ} \times \frac{\alpha\gamma}{2ac}(1+\varrho)^{-n} + \frac{1+\overset{\prime}{A}^{\circ}C^{\circ}}{2(1+\varrho)}\cdot\frac{\overset{\prime}{\alpha}\gamma}{ac}(1+\varrho)^{-n} - \underset{\prime}{A}^{\circ}C^{\circ} \times \frac{\alpha\underset{\prime}{\gamma}}{2ac}(1+\varrho)^{-n}$: and which, being added to the first n terms of the several collateral series above mentioned, will make the total present value of the given sum, when B is the oldest of the three lives, equal to $s\left[\mathbb{K} - x\right]$.

§ 324. *Case* 3. Let C be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above mentioned will be equal to $\mathbb{K} - \frac{1-\varrho A'}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n} - \frac{1-\varrho B'}{2(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n} + \frac{1+AB'}{(1+\varrho)}\cdot\frac{\alpha\beta}{ab}(1+\varrho)^{-n} - \frac{1+\acute{A}B'}{2(1+\varrho)}\cdot\frac{\acute{\alpha}\beta}{ab}(1+\varrho)^{-n} - \frac{1+\acute{A}'\acute{B}'}{2(1+\varrho)}\cdot\frac{\alpha\acute{\beta}}{ab}(1+\varrho)^{-n}$. But, after the decease of C, the chance of receiving the sum at the end of any subsequent year, will depend on the happening of two different events: 1° that only A dies in the year, C having died after B in either of the preceding years (the probability of which latter contingency will now be denoted by ϕ): 2° that only B dies in the year, C having died after A in either of the preceding years (the probability of which latter contingency will now be denoted by ϖ). Consequently, the sum of the expectations on the happening of these events, continued to the utmost extent of human life, will be equal to $s.\phi \times \frac{1-\varrho A'}{(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n} + s.\varpi \times \frac{1\ \ \varrho B'}{(1+\varrho)} \times \frac{\beta}{b}(1+\varrho)^{-n}$: and which, being added to the sum of the first n terms of the several collateral series above mentioned, will make the total present value of the given sum, when C is the oldest life, equal to $s\left[\mathbb{K} - y - z\right]$.*

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* It may not be improper here to remark that the present values of the sums, found by Prob. XLVI, XLVII, and XLVIII, are together equal to $s \times \frac{1-\varrho(A+B-AB)}{(1+\varrho)}$; or to the present value of the given sum after the longest of the two lives A, B.

COROLLARY.

§ 325. When the lives are equal, or all of the same age A, the last three quantities in the formula denoted by $\mathbb{K}$ destroy each other, and the general expression in this case will become $s \times \frac{1-\varrho(3A-3AA+AAA)}{3(1+\varrho)}$: that is, equal to one third of the present value of the given sum to be received on the extinction of the longest of the three lives.

PROBLEM XLIX.*

§ 326. To determine the present value of a given sum payable on the decease of A *and* B, provided B dies before another life C.

SOLUTION.

The payment of the sum at the end of the first year will depend on the happening of either of two events: 1° that all the lives fail in the year, B having died before C: 2° that only A and B die in that year, C living to the end of it. The probabilities of the happening of these two events are respectively $\frac{(a-a').(b-b').(c-c')}{2abc}$ and $\frac{(a-a').(b-b')c'}{abc}$: which being added together, and multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

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* Morgan, Prob. 28; and in Phil. Trans. for 1794, Prob. 3, page 242.

But, in the second and following years the given sum may be received on the happening of either of seven different events: 1° that all the lives fail in the year, B having died before C: 2° that A and B fail in the year, and that C lives to the end of it: 3° that A and C both fail in the year, and that B dies in either of the preceding years: 4° that B and C both fail in the year, B having died first, and that A dies in either of the preceding years: 5° that only A dies in the year, B having failed in either of the preceding years, and C living to the end of it: 6° that only B dies in the year, C living to the end of it, and A having died in either of the preceding years: 7° that only A dies in the year, and *that B and C both fail in either of the preceding years, B having died first*. The probabilities of the happening of these several events in the second year are respectively $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\,(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{c}}{abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2bc}\times\left(1-\frac{\overset{\prime}{a}}{a}\right)$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a})\overset{\prime\prime}{c}}{ac}\left(1-\frac{\overset{\prime}{b}}{b}\right)$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b})\overset{\prime\prime}{c}}{bc}\left(1-\frac{\overset{\prime}{a}}{a}\right)$, and $\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{2a}\left(1-\frac{\overset{\prime}{b}}{b}\right)\cdot\left(1-\frac{\overset{\prime}{c}}{c}\right)$: which, being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner may be found the expectation of receiving the sum at the end of the third and every subsequent year, during the continuance of the oldest life: and if these yearly expectations be reduced to their lowest terms and arranged under each other, they will form fourteen collateral series; the first n terms of which will vary according to the seniority of the lives concerned.

§ 327. *Case* 1. Let A be the oldest of the three lives. In this case the first n terms of the several collateral series above mentioned will be equal to the sum of the terms continued to the utmost extent of human life; because the life of A is involved in each collateral series: therefore its value will be equal to s multiplied into $\frac{1-\varrho A}{2(1+\varrho)} - \frac{1+AB}{2(1+\varrho)} + \frac{1+\overset{\prime}{A}B}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a} + \frac{1+AC}{2(1+\varrho)} - \frac{1+\overset{\prime}{A}C}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a} + \frac{1-\varrho BC}{2(1+\varrho)} - \frac{1+\overset{\prime}{B}C}{2(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{b} + \underset{,}{B}C.\frac{b_{,}}{2b} - \frac{1-\varrho ABC}{2(1+\varrho)} - A\underset{,}{B}C.\frac{b_{,}}{2b} + A\underset{,}{B}C.\frac{c_{,}}{2c}$. But this expression may be reduced to $\frac{1-\varrho(A+BC-ABC)-AB+AC}{2(1+\varrho)} + \frac{(\overset{\prime}{A}B-\overset{\prime}{A}C)\overset{\prime}{a}}{2(1+\varrho)a} - \frac{1}{2b}\left[\frac{(1+\overset{\prime}{B}C)\overset{\prime}{b}}{(1+\varrho)} - (\underset{,}{B}C - A\underset{,}{B}C).b_{,}\right] + A\underset{,}{B}C.\frac{c_{,}}{2c}$: and which I shall denote by $\mathbb{L}$; consequently, when A is the oldest life, the present value required will be represented by $s \times \mathbb{L}$.

§ 328. *Case* 2. Let B be the oldest of the three lives. In this case the first n terms of the several collateral series above mentioned will be equal to $\mathbb{L} - \frac{1-\varrho A^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n} - \frac{1+A^{\circ}C^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha\gamma}{ac}(1+\varrho)^{-n} +$

$\frac{1+\overset{\prime}{A}'C'}{2(1+\varrho)}\cdot\frac{\overset{\prime}{\alpha}\gamma}{ac}(1+\varrho)^{-n}$. But after the decease of B the sum of the expectations for all the subsequent years, continued to the utmost extent of human life, will (as in the second case of Prob. XXXIX) be equal to $s.\phi\times\frac{1-\varrho A'}{(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}$: which being added to the sum of the first n terms of the several collateral series above mentioned, will make the total present value of the given sum equal to $s\Big[\mathbb{L} - x\Big]$.

§ 329. *Case* 3. Let C be the oldest of the three lives. In this case the first n terms of the several collateral series above alluded to will be equal to $\mathbb{L} - \frac{1-\varrho A^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n} + \frac{1+A^{\circ}B^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha\beta}{ab}(1+\varrho)^{-n} - \frac{1+\overset{\prime}{A}^{\circ}B'}{2(1+\varrho)}\cdot\frac{\overset{\prime}{\alpha}\beta}{ab}(1+\varrho)^{-n}$. But, after the decease of C, the sum of the expectations for all the subsequent years may be more correctly expressed as in the preceding case, by $s.\phi\times\frac{1-\varrho A'}{(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}$: and which being added to the first n terms of the several collateral series above mentioned will make the total present value of the given sum, in this case required, equal to $s\Big[\mathbb{L} - z\Big]$.

COROLLARY.

§ 330. When the three lives are equal to each other, or of the same age A, the last three quantities in the formula denoted by 𝕃 destroy each other, and the general expression will in this case become equal to $s \times \frac{1-\varrho(A+AA-AAA)}{2(1+\varrho)}$.[*]

PROBLEM L.[†]

§ 331. To determine the present value of a given sum payable on the decease of A and B, provided another life C dies before B.

SOLUTION.

The payment of the given sum at the end of the first year will depend on the happening of one event only; viz. the extinction of all the lives in that year, C having died before B: the probability of this event is $\frac{(a-\acute{a}).(b-\acute{b}).(c-\acute{c})}{2abc}$; which being multiplied by $s(1+\varrho)^{-1}$, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, the given sum may be received on the happening of either of five different events: 1° that all the lives fail in the year, C having died before B: 2° that A and B fail in the year, and that C fails in either of the preceding years: 3° that B and C both fail in the year, C having died first, and that A dies in either of the preceding years: 4° that only B dies in the year, and that A and C both die in either of the preceding years: 5° that only A fails in the year, and *that B and C both fail in either of the preceding years, B having died last.* The probabilities of the happening of these several events in the second year will be respectively $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2abc}$, $\frac{(\overset{\prime}{a}-\overset{\prime\prime}{a}).(\overset{\prime}{b}-\overset{\prime\prime}{b})}{ab}\left(1-\frac{\overset{\prime}{c}}{c}\right)$, $\frac{(\overset{\prime}{b}-\overset{\prime\prime}{b}).(\overset{\prime}{c}-\overset{\prime\prime}{c})}{2bc}\left(1-\frac{\overset{\prime}{a}}{a}\right)$, $\frac{\overset{\prime}{b}-\overset{\prime\prime}{b}}{b}\times\left(1-\frac{\overset{\prime}{a}}{a}\right)\cdot\left(1-\frac{\overset{\prime}{c}}{c}\right)$, and $\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{2a}\left(1-\frac{\overset{\prime}{b}}{b}\right)\cdot\left(1-\frac{\overset{\prime}{c}}{c}\right)$: which, being added together and multiplied by $s(1+\varrho)^{-2}$, will give the expectation of receiving the sum at the end of the second year. In like manner may be found the expectation of receiving the sum at the end of the third and every subsequent year during the continuance of the oldest life: and, if these several expectations be reduced to their lowest terms and arranged under each other, they will form seventeen collateral series; the sum of the first n terms of which will vary according to the seniority of the lives concerned.

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[*] Mr. Morgan makes the value of the sum in this case equal to $\frac{S(r-1)}{r} \times (V-CC-C-CCC)$; that is, in the present notation, equal to $s \times \frac{1-\varrho(A+AA+AAA)}{(1+\varrho)}$: an error which has also been faithfully transcribed into two successive editions of Dr. Price's treatise. See *Phil. Trans.* for 1794, part II, page 245: and Dr. Price's *Obs. on Rev. Pay.* vol. 1, page 395.

[†] Phil. Trans. for 1794, Prob. 4, page 245; and for 1800, Prob. 7, page 40: which two *might* have been comprised in *one* problem.

§ 332. *Case* 1. Let A be the oldest of the three lives. In this case, the several collateral series here mentioned must be continued to the utmost extent of human life; because the life of A is involved in each perpendicular series: whence, its value will be equal to s multiplied into $\frac{1-\varrho A}{2(1+\varrho)} + \frac{1-\varrho B}{(1+\varrho)} - \frac{1+AB}{2(1+\varrho)} + AB - \frac{1+\overset{\prime}{A}B}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a} - \frac{1+AC}{2(1+\varrho)} + \frac{1+\overset{\prime}{A}C}{2(1+\varrho)}\cdot\frac{\overset{\prime}{a}}{a} - \frac{1-\varrho BC}{2(1+\varrho)} + \frac{1+\overset{\prime}{B}C}{2(1+\varrho)}\cdot\frac{\overset{\prime}{b}}{b} - \underset{\prime}{B}C\cdot\frac{\underset{\prime}{b}}{2b} + \frac{1-\varrho ABC}{2(1+\varrho)} + A\underset{\prime}{B}C\cdot\frac{\underset{\prime}{b}}{2b} - AB\underset{\prime}{C}\cdot\frac{\underset{\prime}{c}}{2c}$. But this expression may be reduced to $\frac{1-\varrho(A+2B-2AB-BC+ABC)+AB-AC}{2(1+\varrho)} - \frac{(\overset{\prime}{A}B-\overset{\prime}{A}C)\overset{\prime}{a}}{2(1+\varrho)a} + \frac{1}{2b}\times\left[\frac{(1+\overset{\prime}{B}C)\overset{\prime}{b}}{(1+\varrho)} - (\underset{\prime}{B}C-A\underset{\prime}{B}C)\underset{\prime}{b}\right] - AB\underset{\prime}{C}\cdot\frac{\underset{\prime}{c}}{2c}$; and which I shall denote by $\mathbb{M}$. Consequently, when A is the oldest life, the required value will be represented by $s \times \mathbb{M}$.

§ 333. *Case* 2. Let B be the oldest of the three lives. In this case, the sum of the first n terms of the several collateral series above alluded to will be equal to $\mathbb{M} - \frac{1-\varrho A^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n} + \frac{1+A^{\circ}C^{\circ}}{2(1+\varrho)}\cdot\frac{\alpha\gamma}{ac} \times (1+\varrho)^{-n} - \frac{1+\overset{\prime}{A}{}^{\circ}C^{\circ}}{2(1+\varrho)}\cdot\frac{\overset{\prime}{\alpha}\gamma}{ac}(1+\varrho)^{-n}$. But after the decease of B, the sum of the expectations for all the subsequent years will be more correctly expressed, as in the second case to Prob. XL, by $s(1-\phi) \times \frac{1-\varrho A^{\circ}}{(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}$; and which, being added to the sum of the first n terms of the several collateral series above mentioned, will make the total present value of the given sum, when B is the oldest life, equal to $s\left[\mathrm{M}+x\right]$.

§ 334. *Case* 3. Let C be the oldest of the three lives. In this case, the first n terms of the several collateral series above alluded to, will be equal to $\mathrm{M}-\frac{1-\varrho A'}{2(1+\varrho)}\cdot\frac{\alpha}{a}(1+\varrho)^{-n}-\frac{1-\varrho B'}{(1+\varrho)}\cdot\frac{\beta}{b}(1+\varrho)^{-n}+\frac{1+A'B'}{2(1+\varrho)}\cdot\frac{\alpha\beta}{ab}(1+\varrho)^{-n}-A'B'.\frac{\alpha\beta}{ab}(1+\varrho)^{-n}+\frac{1+\acute{A}'B'}{2(1+\varrho)}\times\frac{\acute{\alpha}\beta}{ab}(1+\varrho)^{-n}$. But, after the decease of C, the chance of receiving the given sum at the end of any subsequent year will depend on the happening of three different events: 1° that A and B both fail in the year: 2° that only B fails in the year, A having died in either of the preceding years: 3° that only A fails in the year, B having died after C in either of the preceding years. The probabilities of the happening of these several events in the $(n+1)^{\mathrm{st}}$ year are respectively $\frac{(\alpha-\acute{\alpha}).(\beta-\acute{\beta})}{ab}$, $\frac{\beta-\acute{\beta}}{b}\left(1-\frac{\alpha}{a}\right)$ and $\frac{\alpha-\acute{\alpha}}{a}\left(1-\phi-\frac{\beta}{b}\right)$: which, being added together and multiplied by $s(1+\varrho)^{-(n+1)}$, will give the expectation of receiving the sum at the end of the $(n+1)^{\mathrm{st}}$ year.

In like manner we may find the $(n+2)^{\mathrm{nd}}$, and all the subsequent expectations to the utmost extremity of human life: the sum of all which will be equal to

s multiplied into $(1-\phi) \times \frac{1-\varrho A}{2(1+\varrho)} \cdot \frac{\alpha}{a}(1+\varrho)^{-n} + \frac{1-\varrho B}{(1+\varrho)} \cdot \frac{\beta}{b}(1+\varrho)^{-n} - \frac{1+AB}{(1+\varrho)} \cdot \frac{\alpha\beta}{ab}(1+\varrho)^{-n} + AB \times \frac{\alpha\beta}{ab}(1+\varrho)^{-n}$; and which, being added to the first n terms of the several collateral series above found, will make the total present value of the given sum, when C is the oldest life, equal to $s\left[\mathfrak{M} + z\right]$.

COROLLARY.

§ 335. When all the lives are equal, or of the same age A, the last three quantities in the formula denoted by $\mathfrak{M}$ destroy each other, and the general expression in this case will become $s \times \frac{1-\varrho(3A-3AA+AAA)}{2(1+\varrho)}$: that is, equal to half the present value of the given sum, to be received on the extinction of the longest of the three lives.

SCHOLIUM.

§ 336. Since the present values of the sum, as found by the preceding problem, and by the present problem, are together equal to $s \times \frac{1-\varrho(A+B-AB)}{(1+\varrho)}$; or to the present value of the given sum to be received on the extinction of the longest of the two lives A, B: it follows that the present value of one of them being found, the present value of the other may be easily determined by subtracting such value from the general expression here given.

CHAPTER IX.

On M. De Moivre's Hypothesis.

§ 337. If the decrements of life, in any table of observations, were equal and uniform from birth to the utmost extent of human life, the number of persons living at every succeeding age, as shown by that table, would be the terms of a decreasing arithmetical progression. Now, though this fact does not take place throughout the *whole* duration of life; yet, in several of the tables of observations (particularly in the *Northampton* table) it will be seen that, towards the middle ages, the decrements are, for many years together, constant and uniform; or at least very nearly so: and it was upon this view of the subject that M. De Moivre founded his ingenious hypothesis *that the decrements of life are in arithmetical progression.** For, he imagined that the deficiencies in one part of his scale would be compensated by the excesses of the other part: and that though the values deduced from this principle might not be strictly true, yet the error could not be very material. In order to accommodate this hypothesis to the method of determining the value of Life Annuities, it was necessary to suppose the extent of life confined to a certain period of time, which he (for reasons which subsequent observations have shown not to be sufficiently well founded) has taken at 86: remarking, however, at the same time, that the instances of persons living beyond that period are so few as not to be worth notice in such a general view of the subject.

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* The same author also assumed as an hypothesis that the decrements of life were in a *geometrical* ratio: a principle which he applied to the valuation of annuities on *joint* lives. But this theory is too erroneous to be insisted on.

Much obloquy has of late years been thrown on this hypothesis by Dr. Price and Mr. Morgan; the latter of whom has been particularly severe in his comments on its use and application. It is true that more recent discoveries have shown that it cannot *always* be depended upon: and the great share which these gentlemen have had in deducing the values of annuities from *real observations*, and thereby superseding the use of the hypothesis, may be some extenuation for the high tone they have assumed on this occasion. Nevertheless, the hypothesis itself is *still* of great and extensive use in the doctrine of annuities; and will ever remain a monument of the ingenuity and abilities of its illustrious inventor.

§ 338. The reader must be readily aware that the great advantage of M. De Moivre's hypothesis is the remarkable facility which it affords for determining the present value of annuities on any single or joint lives. The method of deducing such values from the first problem of the present work (see page 29) is extremely laborious; since the numerators of the fractions, which form the series there given, do not decrease in any regular manner. But, when such numerators are in arithmetical progression, the sum of the whole series may be expressed in a general formula; and its value may then be found almost as soon as one term only of the other series just alluded to.

Although M. De Moivre, in the application of his principles to practical purposes, has assumed 86 as the extreme period of human life, (that is; that, out of 86 persons born, *one* of them will die annually till they are all extinct) yet this number does not necessarily follow from his hypothesis: for any other number, which might be found more agreeable to real observations, may be substituted in lieu thereof. Assuming, however, the same period as De Moivre, it is evident that the number of persons living at any age will be equal to the number of years between that age and 86: this number he calls the *complement* of life. Consequently a, b, c, &c will, upon this hypothesis, be equal to the complement of the lives A, B, C, &c, respectively: or, equal to the difference between their respective ages and 86.

§ 339. This being premised, it will be readily seen that the expression $\frac{1}{a}\left[\frac{\overset{\prime}{a}}{(1+\varrho)} + \frac{\overset{\prime\prime}{a}}{(1+\varrho)^2} + \frac{\overset{\prime\prime\prime}{a}}{(1+\varrho)^3} + \&c\right]$ which, by Prob. I, cor. 1, denotes the value of an annuity on the life A, will become

$\frac{1}{a}\left[\frac{a-1}{(1+\varrho)}+\frac{a-2}{(1+\varrho)^2}+\frac{a-3}{(1+\varrho)^3}+\dots\dots\frac{a-a}{(1+\varrho)^a}\right]$. Now, this series may be divided into two others, the first of which is $\left[\frac{1}{(1+\varrho)}+\frac{1}{(1+\varrho)^2}+\frac{1}{(1+\varrho)^3}+\dots\dots\frac{1}{(1+\varrho)^a}\right]$; and the second, which is to be subtracted, is $\frac{1}{a}\left[\frac{1}{(1+\varrho)}+\frac{2}{(1+\varrho)^2}+\frac{3}{(1+\varrho)^3}+\dots\dots\frac{a}{(1+\varrho)^a}\right]$. But the former of these, which I shall denote by y, is equal to the present value of an annuity certain for the term a; and the latter is equal to $\frac{1}{a}\times\frac{1+y-(a+1).(1+\varrho)^{-a}}{\varrho}$:* consequently the total present value of the annuity is, upon this hypothesis, equal to $y-\frac{1+y-(a+1).(1+\varrho)^{-a}}{a\varrho}=\frac{a-(1+\varrho)y}{a\varrho}$.†

§ 340. In like manner, the expression $\frac{1}{ab}\left[\frac{a'b'}{(1+\varrho)}+\frac{a''b''}{(1+\varrho)^2}+\frac{a'''b'''}{(1+\varrho)^3}+\&c\right]$, which denotes the value of an annuity on the two joint lives A B, will become, upon this hypothesis, equal to $\frac{1}{ab}\left[\frac{(a-1).(b-1)}{(1+\varrho)}+\frac{(a-2).(b-2)}{(1+\varrho)^2}+\frac{(a-3).(b-3)}{(1+\varrho)^3}+\dots\dots\frac{(a-b).(b-b)}{(1+\varrho)^b}\right]$;

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* See my *Doctrine of Interest and Annuities*, page 89: where the method of summing this and the following series is pointed out.

† See Dodson, vol. ii. Ques. 56. Simpson, Prob. 1, cor. 5. De Moivre, page 310.

a series which is to be continued for b terms only, B being the oldest of the two lives. But, if the numerators of the fractions composing this series be expanded by multiplication, the whole expression may be divided into three other distinct series: viz.

$$\left[\frac{1}{(1+\varrho)} + \frac{1}{(1+\varrho)^2} + \frac{1}{(1+\varrho)^3} + \ldots\ldots \frac{1}{(1+\varrho)^b}\right] -$$
$$\frac{a+b}{ab}\left[\frac{1}{(1+\varrho)} + \frac{2}{(1+\varrho)^2} + \frac{3}{(1+\varrho)^3} + \ldots\ldots \frac{b}{(1+\varrho)^b}\right]$$
$$+ \frac{1}{ab}\left[\frac{1}{(1+\varrho)} + \frac{4}{(1+\varrho)^2} + \frac{9}{(1+\varrho)^3} + \ldots\ldots \frac{b^2}{(1+\varrho)^b}\right].$$

The first of these is equal to the present value of an annuity for the term b, and I shall denote it by y: the second is (from what has been said above) equal to $-\frac{a+b}{ab} \times \frac{1+y-(b+1).(1+\varrho)^{-b}}{\varrho}$: and the third is equal to $\frac{1}{ab\varrho}\left[1+y+2 \times \frac{1+y-(b+1).(1+\varrho)^{-b}}{\varrho} - (b+1)^2 \times (1+\varrho)^{-b}\right]$: consequently the total present value of the annuity for two joint lives is equal to $\frac{1}{\varrho} - \frac{1+\varrho}{a\varrho} \times \left[\frac{y}{b}\left(a-b-1-\frac{2}{\varrho}\right)+\frac{2}{\varrho}\right]$.*

§ 341. By a similar method of reasoning it will be found that the expression $\frac{1}{abc}\left[\frac{a'b'c'}{(1+\varrho)} + \frac{a''b''c''}{(1+\varrho)^2} + \frac{a'''b'''c'''}{(1+\varrho)^3} + \&c\right]$, which denotes the value of an annuity on the three joint lives A B C, will become, upon this hypothesis, equal to $\frac{1}{abc}\left[\frac{(a-1).(b-1).(c-1)}{(1+\varrho)} + \frac{(a-2).(b-2).(c-2)}{(1+\varrho)^2} + \frac{(a-3).(b-3).(c-3)}{(1+\varrho)^3} + \ldots\ldots\ldots \frac{(a-c).(b-c).(c-c)}{(1+\varrho)^c}\right]$; a series which is to be continued for c terms only, C being now the oldest life. But the whole of this series may, by expansion, be converted into the four following series: viz. $\left[\frac{1}{(1+\varrho)} + \frac{1}{(1+\varrho)^2} + \frac{1}{(1+\varrho)^3} + \ldots\ldots\ldots \frac{1}{(1+\varrho)^c}\right] - \frac{ab+ac+bc}{abc} \times \left[\frac{1}{(1+\varrho)} + \frac{2}{(1+\varrho)^2} + \frac{3}{(1+\varrho)^3} + \ldots\ldots\ldots \frac{c}{(1+\varrho)^c}\right] + \frac{a+b+c}{abc}\left[\frac{1}{(1+\varrho)} + \frac{4}{(1+\varrho)^2} + \frac{9}{(1+\varrho)^3} + \ldots\ldots\ldots \frac{c^2}{(1+\varrho)^c}\right] - \frac{1}{abc}\left[\frac{1}{(1+\varrho)} + \frac{8}{(1+\varrho)^2} + \frac{27}{(1+\varrho)^3} + \ldots\ldots\ldots \frac{c^3}{(1+\varrho)^c}\right]$. Now, the first of these series is equal to the present value of an annuity certain for the term c; and I shall denote it by y; and the remaining ones may be summed in the same manner

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* See Dodson, vol. ii. Ques. 64. Simpson, Prob. 1, cor. 5. Price, note (L). I shall here take the opportunity of observing that the strictures of Dr. Price on M. De Moivre's *rules* for calcu-

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lating the value of annuities on joint lives (inserted in the 4th Chapter of his *Obs. on Rev. Pay.*) allude altogether to an *approximation* given by De Moivre as deduced from his hypothesis of a *geometrical* ratio, and not to the *correct* expression. The formula in the text will oftentimes give the same values as those deduced from the *Northampton* Table; as may be seen by a comparison with the Tables at the end of the present work: and I shall show in the sequel that these formulæ may be used with good effect in determining, with a tolerable degree of correctness, the value of annuities deduced from the *real probabilities* of life. See § 346.

as in the preceding cases: whence the total present value of an annuity on the three joint lives is equal to $\frac{1}{\varrho} - \frac{1+\varrho}{ab\varrho}\Big[\frac{1}{\varrho}\Big(2b + 2a - c - 3 - \frac{6}{\varrho}\Big) - \frac{2y}{c\varrho}\Big(b + a - 2c - 3 - \frac{3}{\varrho}\Big) + \frac{y}{c}(b - c - 1).(a - c - 1)\Big]$.[*]

§ 342. But, since the publication of so many accurate tables of the values of annuities, deduced from *real observations*, these formulæ have become of little or no use; and are seldom resorted to, unless it is required to find a near value of an annuity on any given lives whose ages are not inserted in those tables. Nevertheless the hypothesis itself is still of great use in the doctrine of annuities; and facilitates very much several calculations arising from this subject: particularly in those cases where the contingency continues for a given *term* only; as will evidently appear from the following problem.

PROBLEM LI.[†]

§ 343. To find the value of an annuity on a given life, during part of whose existence the decrements of life may be considered as equal.

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[*] See Dodson, vol. ii. Ques. 69. Simpson, Prob. 1, cor. 5.

[†] De Moivre, page 341. Simpson, Prob. 1, cor. 8. Price, note (N). Dodson, vol. ii. Ques. 68. This last author has also given a method of solution (Question 61) provided the decrements of life be divided into *several* arithmetical progressions, each differing from one another.

SOLUTION.

Let A be the given life, and n the interval of equal decrements: also let a be the number of persons living at the age of A, and α the number of persons living at an age n years older; and let the uniform decrements of life be denoted by δ. Then will the value of an annuity on this life be denoted by the series $\frac{1}{a}\left[\frac{a-\delta}{(1+\varrho)}+\frac{a-2\delta}{(1+\varrho)^2}+\frac{a-3\delta}{(1+\varrho)^3}+\ldots\ldots\frac{a-n\delta}{(1+\varrho)^n}+\frac{\overset{\prime}{\alpha}}{(1+\varrho)^{n+1}}+\frac{\overset{\prime\prime}{\alpha}}{(1+\varrho)^{n+2}}+\frac{\overset{\prime\prime}{\alpha}}{(1+\varrho)^{n+3}}+\&c\right]$; which may be divided into two parts: viz. the series $\frac{1}{a}\left[\frac{a-\delta}{(1+\varrho)}+\frac{a-2\delta}{(1+\varrho)^2}+\frac{a-3\delta}{(1+\varrho)^3}+\ldots\ldots\ldots\frac{a-n\delta}{(1+\varrho)^n}\right]$ $+\frac{1}{a}\left[\frac{\overset{\prime}{\alpha}}{(1+\varrho)^{n+1}}+\frac{\overset{\prime\prime}{\alpha}}{(1+\varrho)^{n+2}}+\frac{\overset{\prime\prime\prime}{\alpha}}{(1+\varrho)^{n+3}}+\&c\right]$. The latter of these is, by Prob. I, cor. 3, equal to $(A)^{\mathrm{d}}$, or equal to the value of an annuity on the life A *deferred* for n years: and the former may be divided into two others; viz. $\left[\frac{1}{(1+\varrho)}+\frac{1}{(1+\varrho)^2}+\frac{1}{(1+\varrho)^3}+\ldots\ldots\frac{1}{(1+\varrho)^n}\right]-\frac{\delta}{a}\left[\frac{1}{(1+\varrho)}+\frac{2}{(1+\varrho)^2}+\frac{3}{(1+\varrho)^3}+\ldots\ldots\frac{n}{(1+\varrho)^n}\right]$: the sum of which may be readily found, from what has preceded, to be equal to $y-\frac{\delta}{a\varrho}\left[(1+\varrho)y-n(1+\varrho)^{-n}\right]$. Consequently the total value of the annuity will be $y-\frac{\delta}{a\varrho}\Big[(1+\varrho)y-$

$n(1+\varrho)^{-n}\Big] + (A)^{\mathrm{d}}$. But, since δ is always equal to $\frac{a-\alpha}{n}$;* and since $(A)^{\mathrm{d}}$ is equal to $A' \times \frac{\alpha}{a}(1+\varrho)^{-n}$; this expression may be rendered more convenient for practice by making those substitutions: whence the value will become $\frac{1}{\varrho} - \frac{(a-\alpha).(1+\varrho)y}{an\varrho} - \frac{\alpha(1+\varrho)^{-n}}{a\varrho} + A' \times \frac{\alpha}{a}(1+\varrho)^{-n} = \frac{1}{\varrho} - \frac{1}{a}\Big[\frac{a-\alpha}{n\varrho}(1+\varrho)y - \alpha\big(\frac{1}{\varrho} - A'\big).(1+\varrho)^{-n}\Big]$. Now it appears by the *Northampton* table of observations that the decrements of life are *nearly* uniform during the whole interval from the age of 20 to the age of 80: if therefore the value of an annuity on a life aged 80 be once calculated, the value of an annuity on a life of any intermediate age may be easily deduced from the formula here given; and will agree very nearly with the true values.

Example. Let the rate of interest be 5 per cent; in which case the value of an annuity on a life of 80 years of age will be 3·515: and, if from this we wish to deduce the value of an annuity on a life of 20 years of age, the formula will become $20 - \frac{1}{5132}\Big[\frac{5132-469}{60 \times 05} \times 1{\cdot}05 \times 18{\cdot}9293 - 469 \times (20 - 3{\cdot}515) \times {\cdot}0535\Big] = 14{\cdot}061$: and which is very near 14·007, or the true value as shown by the tables.

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* This may generally be assumed as the true value, without material error, even when the decrements are not exactly regular.

Now, when it is considered that the value, here found, could not have been deduced from the corollary in page 29, without the actual calculation of *sixty* terms of the series there given, the utility of this formula will be manifest.

COROLLARY.

§ 344. If the value of the annuity is required for a given *term* $(=n)$; and that term happens to be wholly within the interval of equal decrements as shown by any given table of observations; the series $\frac{1}{a}\left[\frac{a-\delta}{(1+\varrho)}+\frac{a-2\delta}{(1+\varrho)^2}+\frac{a-3\delta}{(1+\varrho)^3}+\ldots\ldots\frac{a-n\delta}{(1+\varrho)^n}\right]$ $=y-\frac{a-\alpha}{an\varrho}\left[(1+\varrho)y-n(1+\varrho)^{-n}\right]$ will denote the *exact* value in such case; and is a formula of considerable utility when we are not possessed of any tables of the value of annuities deduced from such observations. Or, if the decrements are very nearly equal, the formula will not differ materially from the true expression.

Example. Suppose it were required to find the value of an annuity for 20 years on a life aged 20, reckoning interest at 5 per cent and the probabilities of living as at *Northampton*. In this case the formula would become $12{\cdot}4622-\frac{5132-3635}{5132\times20\times{\cdot}05}\times\left[1{\cdot}05\times 12{\cdot}4622-20\times{\cdot}3769\right]=10{\cdot}844$: which is the *correct* value of a temporary annuity for 20 years on a life aged 20; because it will be seen, by Table XXV, that from the age of 20 to the age of 40 the decrements of life are equal. The value, deduced from the rule in page 46, is 10·847.

§ 345. This problem and its corollary will serve to show the useful purposes to which M. De Moivre's hypothesis may be occasionally appropriated, and the method of applying it whenever an opportunity occurs. But since the decrements of life are most irregular in the younger and in the latter periods of existence, and are uniform (or nearly so) during the middle ages only; it will be found that this hypothesis cannot in all cases be safely used, unless in deducing the value of annuities or assurances for *terms*. In this respect it is of singular utility, and will be often found to save a laborious calculation, as I have already pointed out in the notes to some of the preceding problems: and, therefore, it will be unnecessary to enlarge more upon the subject in this place. It is in this manner that Mr. Morgan has condescended to use it: but it is done clandestinely, and (I know not for what reason) by a previous *denial* of the fact.*

I would here observe that the rule for determining the value of such annuities as depend on the whole continuance of any number of lives out of any other number of lives, or such as are in reversion or depending upon survivorships, and in general for determining such problems as are contained in the second, third and fourth Chapters of this work are the same on M. De Moivre's hypothesis, as where they are deduced from real observations. For, in each case, the solutions to such questions are obtained from *Tables* which show the values of annuities on single and joint lives: and therefore the merit or demerit of M. De Moivre's hypothesis, as far as regards the value of *annuities*, will rest on the fundamental propositions above given.

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* See Price's *Obs. on Rev. Pay.* vol. 1, page 61, note e: also the note in page 192 of the present work.

But in deducing the value of *assurances*, or reversionary sums, his rules are certainly much more simple than when deduced from real observations: and it would be a fortunate circumstance if his hypothesis could be depended upon throughout the *whole duration* of life. As this, however, is not the case, we must be content with the facility which it oftentimes affords us of determining, in many cases, a very near value of such assurances for *terms:* as I have already fully explained in the sixth and eighth Chapters.*

On the Method of approximating to the Value of Life Annuities.

§ 346. I cannot dismiss this chapter, however, without noticing the utility and convenience of the *formulæ* arising from M. De Moivre's hypothesis, in enabling us also to deduce (from the values of annuities on single or joint lives at any one rate of interest) the values of annuities on the same lives, at any *other* rate of interest. In order to explain this method, I would observe that, according to M. De Moivre's hypothesis, the *expectation* of any life is equal to half the complement of such life:* consequently, the complement of any life is equal to twice its expectation. If, therefore, we substitute *twice the expectation* of any life deduced from real observations, instead of the quantities *a*, *b*, or *c*, in the general formulæ in pages 316, 317 and 319, the values thence arising will in most cases be much nearer the true values of annuities deduced from such observations, than when *a*, *b*, or *c*, is taken equal to the complement of such life according to M. De Moivre's hypothesis.† Or, (which is all that is required in the present instance) the difference between the values of an annuity on any single or joint lives, deduced in this manner from the *expectations* of life, at any two rates of interest, will be nearly the same as the difference between the *correct* values of a similar annuity, at the same rates of interest, deduced from *real observations*. Consequently, when the value of an annuity according to any one rate of interest is given, we may readily obtain a very near value of a similar annuity at any other rate, by means of the first difference here alluded to: as will be evident from the following general rule.

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* See the notes in pages 150, 192, 205 and 213.

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* See the note in page 50.

† See the observations at the end of the following note.

Call the *correct* value, computed at any one rate of interest, the FIRST value.

Call the value deduced from the *expectations* of life, at the *same* rate of interest, the SECOND value.

Call the value deduced from the *expectations* of life, at any *other* rate of interest, the THIRD value.

Then, the *difference* between the SECOND and THIRD values subtracted from, or added to, the FIRST value (according as the SECOND is greater or less than the THIRD) will be the near value of the annuity at the *other* rate of interest required.

Example 1. What is the near value of an annuity on a life aged 20 years at 4 per cent interest, deduced from the correct value at 5 per cent, and according to the *Northampton* observations?

The *first*, or correct value at 5 per cent is, by Table XXVII, equal to 14·007. The *second* value, deduced at the same rate of interest from the expectation of that life by the formula in page 316, is equal to $\frac{66{\cdot}86 - 1{\cdot}05 \times 19{\cdot}234}{66{\cdot}86 \times {\cdot}05} = 13{\cdot}959$.* The *third* value, deduced from the same formula, at 4 per cent, is equal to $\frac{66{\cdot}86-1{\cdot}04\times 23{\cdot}183}{66{\cdot}86\times {\cdot}04}=15{\cdot}984$. Therefore $14{\cdot}007+(15{\cdot}984-13{\cdot}959)=16{\cdot}032$ will be the near value required: and which differs only an unit in the last figure from the true value as given in Table XXVII.

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* When twice the expectation is equal to a whole number with a decimal added (as is commonly the case), the value of an annuity for that term may be best computed in the following manner. Suppose the number of years (as in the present case) to be 66·86. The value of an annuity for 66 years is, by Table LIX, equal to 19·201; and the value of an annuity for 67 years is 19·239. The difference between these two values is ·038; which, being multiplied by the decimal ·86, and the product ·033 added to the least

§ 347. The same principles will apply to the case of two *joint* lives: and it will be found that in both cases the deduced values are sometimes nearly the same as the correct values; that generally they do not differ more than a 20th or 30th part of a year's purchase; that in *joint* lives they differ less than in single lives; and that they come equally near to each other *whatever* the rates of interest are.*

of the two values, will give 19·234 for the value of the annuity for 66·86 years.

The second and third values here obtained (that is, 13·959 and 15·984) will be found, on a comparison with the values in Table XXVII, to be much nearer the *true* values than those obtained from M. De Moivre's hypothesis, as given in Table XLIX. Consequently this first step of the process will show that M. De Moivre's *formula* (as given in page 316) may sometimes be applied, with good effect, to find, in an expeditious manner by *one* operation, a near value of an annuity deduced from real observations.

* See a variety of examples, in proof of these assertions, in Dr. Price's *Obs. on Rev. Pay.* vol. 1, page 231. The same author remarks that "these deductions, in the case of single lives parti-"cularly, are so easy, and give the true values so nearly, that it "will be scarcely ever *necessary* to calculate the exact values (ac-"cording to any given observations) for more than *one* rate of "interest." But, however convenient the above rules may be in our present dearth of useful Tables, they by no means remove

On the Value of increasing Life Annuities.

§ 348. The hypothesis of M. De Moivre furnishes us likewise with a convenient and useful formula for determining the value of *increasing* Life Annuities: that is, of £1, £2, £3, &c (or any multiples of those sums) payable at the end of 1, 2, 3, &c, years respectively, if the given life A be then in existence. For, the series expressing such value will (from what has been said in page 316) be evidently equal to $\frac{1}{a}\left[\frac{a-1}{(1+\varrho)}+\frac{2(a-2)}{(1+\varrho)^2}+\frac{3(a-3)}{(1+\varrho)^3}+\ldots\ldots\frac{a(a-a)}{(1+\varrho)^a}\right]$, and which may be divided into the two following series,

$$\left[\frac{1}{(1+\varrho)}+\frac{2}{(1+\varrho)^2}+\frac{3}{(1+\varrho)^3}+\ldots\ldots\ldots\frac{a}{(1+\varrho)^a}\right]-$$
$$\frac{1}{a}\left[\frac{1^2}{(1+\varrho)}+\frac{2^2}{(1+\varrho)^2}+\frac{3^2}{(1+\varrho)^3}+\ldots\ldots\frac{a^2}{(1+\varrho)^a}\right].$$

In order to abridge the subsequent process, let us make

$$\left[\frac{1}{(1+\varrho)}+\frac{1}{(1+\varrho)^2}+\frac{1}{(1+\varrho)^3}+\ldots\ldots\ldots\frac{1}{(1+\varrho)^a}\right]=y$$
$$\left[\frac{1}{(1+\varrho)}+\frac{2}{(1+\varrho)^2}+\frac{3}{(1+\varrho)^3}+\ldots\ldots\ldots\frac{a}{(1+\varrho)^a}\right]=x$$
$$\left[\frac{1^2}{(1+\varrho)}+\frac{2^2}{(1+\varrho)^2}+\frac{3^2}{(1+\varrho)^3}+\ldots\ldots\ldots\frac{a^2}{(1+\varrho)^a}\right]=w$$

then will the sum of the two series above given, or the value of the annuity required, be denoted by $x-\frac{w}{a}$.

the necessity of calculating, at *several* rates of interest, any new Tables that may be hereafter formed: and I should hope that no one, who may at any future time undertake this laborious task, will be influenced by so weak and so ill-judged an excuse. The object and real utility of Tables, of any kind, is to save time and labour, and to prevent the occurrence of errors.

But, from what has been said in § 339, it will be seen that the value of an annuity on a single life A is denoted by $y-\frac{x}{a}$; and, from what has been said in § 340, that the value of an annuity on two equal joint lives AA is denoted by $y-\frac{2x}{a}+\frac{w}{aa}$. Therefore $A-AA=y-\frac{x}{a}-y+\frac{2x}{a}-\frac{w}{aa}=\frac{x}{a}-\frac{w}{aa}$; consequently $a(A-AA)=x-\frac{w}{a}$ will be the value of the increasing annuity required: whence we deduce the following rule.

§ 349. *From the value of an annuity on the given life subtract the value of an annuity on two equal joint lives, of the same age with the given life; multiply the difference by the complement (or twice the expectation) of the given life: and the product will be the answer required.*

§ 350. *Example.* Let the given life be 40 years of age; and let the annuity be £1 the first year, £2 the second year, £3 the third year, and so on according to the order of the natural numbers: what is the present value of this annuity, reckoning interest at 4 per cent, and the probabilities of living as observed at *Northampton?*

Here we shall have $A=13{\cdot}197$, $AA=9{\cdot}820$, and a (or twice the expectation) $=46{\cdot}16$: consequently $46{\cdot}16\times(13{\cdot}197-9{\cdot}820)=155{\cdot}882$ will be the value required.

If the annuities had been £10, £20, £30, &c, the present value would be 155·882 × 10. Or, if they had been £15, £30, £45, &c, the present value would be 155·882 × 15.

But, if the annuity *commences* with a larger sum than £1, and yet increases only by £1 in every year, we must add to the value, above found, the value of an annuity on the given life multiplied into the first payment lessened by unity; and the sum will be the answer. Thus, if the annuities in the first case above mentioned, had been £15, £16, £17, &c, we must multiply 13·197 by 14; and the product, or 184·758, added to 155·882 will give 340·640 for the answer in this case required.

§ 351. These and many other instances (in addition to those already mentioned in various parts of this treatise) might be adduced to show the great utility and convenience of M. De Moivre's hypothesis in a general point of view. The most common cases will convince us that it may always lay claim to a considerable share of merit; but that it is particularly entitled to our approbation in enabling us to conduct our inquiries into many branches of this science where the common analysis is not only exceedingly intricate but sometimes entirely fails: and that it is by no means deserving of the false and ignominious epithets of "wretched" or "absurd."

CHAPTER X.

On the Value of Annuities payable Half-yearly, &c. On Half-yearly &c Assurances; and on Annuities secured by Land.

§ 352. In the preceding chapters, the values of annuities have been deduced on the supposition that they are all payable yearly: this is the most usual case. But, as others may occasionally occur, it will be useful to know the limits of the differences which arise in those cases; therefore, that nothing might be wanting on this subject, I shall make no apology for introducing the following investigations.*

If a, $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, $\overset{\prime\prime\prime}{a}$, &c represent the number of persons living at the age of A, and at the age of 1, 2, 3, &c years older than A, agreeably to what has been said in § 23, then will $\frac{a+\overset{\prime}{a}}{2}$, $\frac{\overset{\prime}{a}+\overset{\prime\prime}{a}}{2}$, $\frac{\overset{\prime\prime}{a}+\overset{\prime\prime\prime}{a}}{2}$, &c denote the number of persons living at the end of $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, &c years from the age of A: which, though perhaps not in *all* cases strictly true, will serve our present purpose, and be as near the real value as we could hope for. Consequently, the present value of an annuity on the life A, payable *Half-yearly*, is equal to $\frac{1}{2a}\left[\frac{a+\overset{\prime}{a}}{2(1+\varrho)^{\frac{1}{2}}} + \frac{\overset{\prime}{a}}{(1+\varrho)} + \frac{\overset{\prime}{a}+\overset{\prime\prime}{a}}{2(1+\varrho)^{\frac{3}{2}}} + \frac{\overset{\prime\prime}{a}}{(1+\varrho)^2} + \frac{\overset{\prime\prime}{a}+\overset{\prime\prime\prime}{a}}{2(1+\varrho)^{\frac{5}{2}}} + \frac{\overset{\prime\prime\prime}{a}}{(1+\varrho)^3} + \&c\right]$:* a series which may evidently be divided into the two following ones; viz. $\frac{1}{2a}\left[\frac{a+\overset{\prime}{a}}{2(1+\varrho)^{\frac{1}{2}}} + \frac{\overset{\prime}{a}+\overset{\prime\prime}{a}}{2(1+\varrho)^{\frac{3}{2}}} + \frac{\overset{\prime\prime}{a}+\overset{\prime\prime\prime}{a}}{2(1+\varrho)^{\frac{5}{2}}} + \&c\right] + \frac{1}{2a}\left[\frac{\overset{\prime}{a}}{(1+\varrho)} + \frac{\overset{\prime\prime}{a}}{(1+\varrho)^2} + \frac{\overset{\prime\prime\prime}{a}}{(1+\varrho)^3} + \&c\right]$. The latter of these is equal to $\frac{A}{2}$; and the former, which may be divided again into the two following series $\frac{1}{4a(1+\varrho)^{\frac{1}{2}}}\left[a + \frac{\overset{\prime}{a}}{(1+\varrho)} + \frac{\overset{\prime\prime}{a}}{(1+\varrho)^2} + \frac{\overset{\prime\prime\prime}{a}}{(1+\varrho)^3} + \&c\right] + \frac{(1+\varrho)^{\frac{1}{2}}}{4a}\left[\frac{\overset{\prime}{a}}{(1+\varrho)} + \frac{\overset{\prime\prime}{a}}{(1+\varrho)^2} + \frac{\overset{\prime\prime\prime}{a}}{(1+\varrho)^3} + \&c\right]$, is equal to $\frac{1+A}{4(1+\varrho)^{\frac{1}{2}}} + \frac{(1+\varrho)^{\frac{1}{2}}A}{4}$. Whence

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* A person who receives a life annuity *half-yearly* has a double advantage over one who receives the same annuity yearly: for, besides the interest of each half-yearly payment for six months, he has a chance of receiving one half year's payment *more* than if he were paid yearly. In like manner, a person who receives a life annuity *quarterly* has a double advantage over one who receives the same annuity half-yearly: &c, &c. See this subject detailed at full length in Baron Maseres's *Doctrine of Life Annuities*, pages 233—260.

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[*] The reader is supposed to be acquainted with the method of deducing the present value of an annuity *certain* payable half-yearly &c, as explained in my *Doctrine of Interest and Annuities*, Chapter X.

the total present value of the annuity is equal to $\frac{A}{2}+\frac{1+A}{4(1+\varrho)^{\frac{1}{2}}}+\frac{(1+\varrho)^{\frac{1}{2}}A}{4}=\frac{2(1+\varrho)^{\frac{1}{2}}A+1+A+(1+\varrho)A}{4(1+\varrho)^{\frac{1}{2}}}=\frac{2(1+\varrho)^{\frac{1}{2}}+2+\varrho}{4(1+\varrho)^{\frac{1}{2}}}\times A+\frac{1}{4(1+\varrho)^{\frac{1}{2}}}$.

But, since the quantity $\frac{2(1+\varrho)^{\frac{1}{2}}+2+\varrho}{4(1+\varrho)^{\frac{1}{2}}}$ seldom much exceeds unity,* this expression may be taken (without material error) equal to $A+\frac{1}{4(1+\varrho)^{\frac{1}{2}}}$; and, since $\frac{1}{4(1+\varrho)^{\frac{1}{2}}}$ is seldom much below $\frac{1}{4}$,† the expression may be still be further reduced to $A+\frac{1}{4}$. That is, if to the value of the annuity payable yearly we add a *quarter of a year's purchase*, the sum will be *very near* the value of the same annuity payable *Half-yearly*. The *exact* values however may be easily determined from the general expression above given.

§ 353. If we wish to determine the present value of a similar annuity payable *Quarterly*, we must take

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* When the rate of interest is 2 per cent per annum, the quantity, here alluded to, is equal to 1·000025; and when the rate of interest is 10 per cent per annum, it is equal to 1·000569: whence a judgment may be formed of its value at any intermediate rate.

† When the rate of interest is 2 per cent per annum, this quantity is equal to ·2475; and, when the rate of interest is 10 per cent per annum, it is equal to ·2384: but, it will be seen that as this quantity *decreases*, the one mentioned in the last note *increases:* whence $A+\frac{1}{4}$ will seldom much exceed the true value.

$\frac{3a+a'}{4}$, $\frac{a+3a'}{4}$, $\frac{3a'+a''}{4}$, $\frac{a'+3a''}{4}$, $\frac{3a''+a'''}{4}$, $\frac{a''+3a'''}{4}$, &c to denote the number of persons living at the end of $\frac{1}{4}$, $\frac{3}{4}$, $1\frac{1}{4}$, $1\frac{3}{4}$, $2\frac{1}{4}$, $2\frac{3}{4}$, &c years:* whence, the present value of an annuity on the life A payable *Quarterly* will be equal to

$$\frac{1}{4a}\left[\frac{3a+a'}{4(1+\varrho)^{\frac{1}{4}}}+\frac{a+a'}{2(1+\varrho)^{\frac{1}{2}}}+\frac{a+3a'}{4(1+\varrho)^{\frac{3}{4}}}+\frac{a'}{(1+\varrho)}+\frac{3a'+a''}{4(1+\varrho)^{\frac{5}{4}}}+\frac{a'+a''}{2(1+\varrho)^{\frac{3}{2}}}+\frac{a'+3a''}{4(1+\varrho)^{\frac{7}{4}}}+\frac{a''}{(1+\varrho)^{2}}+\&c\right].$$

But, this series may be divided into the four following ones: viz.

$$\frac{1}{16a(1+\varrho)^{\frac{1}{4}}}\left[\frac{3a+a'}{1}+\frac{3a'+a''}{(1+\varrho)}+\frac{3a''+a'''}{(1+\varrho)^{2}}+\&c\right]=\frac{3(1+A)}{16(1+\varrho)^{\frac{1}{4}}}+\frac{A(1+\varrho)^{\frac{3}{4}}}{16};$$

$$\frac{1}{8a(1+\varrho)^{\frac{1}{2}}}\left[\frac{a+a'}{1}+\frac{a'+a''}{(1+\varrho)}+\frac{a''+a'''}{(1+\varrho)^{3}}+\&c\right]=\frac{1+A}{8(1+\varrho)^{\frac{1}{2}}}+\frac{A(1+\varrho)^{\frac{1}{2}}}{8};$$

$$\frac{1}{16a(1+\varrho)^{\frac{3}{4}}}\times\left[\frac{a+3a'}{1}+\frac{a'+3a''}{(1+\varrho)}+\frac{a''+3a'''}{(1+\varrho)^{2}}+\&c\right]=\frac{1+A}{16(1+\varrho)^{\frac{3}{4}}}+\frac{3A(1+\varrho)^{\frac{1}{4}}}{16};$$

and

$$\frac{1}{4a}\left[\frac{a'}{(1+\varrho)}+\frac{a''}{(1+\varrho)^{2}}+\frac{a'''}{(1+\varrho)^{3}}+\&c\right]=\frac{A}{4}.$$

Whence, the total present value of the annuity is equal to

$$\frac{3(1+A)}{16(1+\varrho)^{\frac{1}{4}}}+\frac{A(1+\varrho)^{\frac{3}{4}}}{16}+\frac{1+A}{8(1+\varrho)^{\frac{1}{2}}}+\frac{A(1+\varrho)^{\frac{1}{2}}}{8}+\frac{1+A}{16(1+\varrho)^{\frac{3}{4}}}+\frac{3A(1+\varrho)^{\frac{1}{4}}}{16}+\frac{A}{4}=$$

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* These are the arithmetical means between the number of persons living at the age of A and at the several ages of $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, &c, years older: and are sufficiently near for the purposes here intended.

$$\frac{4(1+\varrho)^{\frac{3}{4}}+(4+\varrho).(1+\varrho)^{\frac{1}{2}}+2(2+\varrho).(1+\varrho)^{\frac{1}{4}}+4+3\varrho}{16(1+\varrho)^{\frac{3}{4}}}\times A+$$
$$\frac{3(1+\varrho)^{\frac{1}{2}}+2(1+\varrho)^{\frac{1}{4}}+1}{16(1+\varrho)^{\frac{3}{4}}}.$$

But, since the fractional quantity, by which *A* is multiplied, seldom much exceeds unity,* this expression may be taken (without material error) equal to $A+\frac{3(1+\varrho)^{\frac{1}{2}}+2(1+\varrho)^{\frac{1}{4}}+1}{16(1+\varrho)^{\frac{3}{4}}}$: and since $\frac{3(1+\varrho)^{\frac{1}{2}}+2(1+\varrho)^{\frac{1}{4}}+1}{16(1+\varrho)^{\frac{3}{4}}}$ is seldom much below $\frac{3}{8}$,† the expression may be still further reduced to $A+\frac{3}{8}$. That is, if to the value of the annuity payable yearly we add *three eighths of a year's purchase*, the sum will be *very near* the value of the same annuity payable *Quarterly*. The exact values, however, may be easily determined, as in the former case, from the general expression above given.

§ 354. Upon M. De Moivre's hypothesis the present value of a life annuity payable *Half-yearly* will be denoted by the series $\frac{1}{2a}\left[\frac{a-\frac{1}{2}}{(1+\varrho)^{\frac{1}{2}}}+\frac{a-1}{(1+\varrho)}+\right.$

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[*] When the rate of interest is 2 per cent per annum, the quantity, here alluded to, is equal to 1·000037; and when the rate of interest is 10 per cent per annum, it is equal to 1·000710; whence a tolerably correct opinion may be formed of its value at any intermediate rate.

[†] When the rate of interest is 2 per cent per annum, this quantity is equal to ·3719; and when the rate of interest is 10 per cent per annum, it is equal to ·3605: but, as this value decreases the one mentioned in the last note increases; whence $A+\frac{3}{8}$ will seldom much exceed the true value.

$$\frac{a-\frac{3}{2}}{(1+\varrho)^{\frac{3}{2}}}+\frac{a-2}{(1+\varrho)^{2}}+\frac{a-\frac{5}{2}}{(1+\varrho)^{\frac{5}{2}}}+\frac{a-3}{(1+\varrho)^{3}}+\ldots\ldots\frac{a-a}{(1+\varrho)^{a}}\Bigg]$$

: which may be divided into the two following ones: viz.

$$\frac{1}{2}\Bigg[\frac{1}{(1+\varrho)^{\frac{1}{2}}}+\frac{1}{(1+\varrho)}+\frac{1}{(1+\varrho)^{\frac{3}{2}}}+\frac{1}{(1+\varrho)^{2}}+\ldots\ldots\frac{a}{(1+\varrho)^{a}}\Bigg]-\frac{1}{4a}\Bigg[\frac{1}{(1+\varrho)^{\frac{1}{2}}}+\frac{2}{(1+\varrho)}+\frac{3}{(1+\varrho)^{\frac{3}{2}}}+\frac{4}{(1+\varrho)^{2}}+\ldots\ldots\frac{2a}{(1+\varrho)^{a}}\Bigg].$$

The first of these is equal to $\frac{1}{2}\times\frac{1-(1+\varrho)^{-a}}{(1+\varrho)^{\frac{1}{2}}-1}=h$;* and the second is equal to $-\frac{1}{4a}\times\frac{2h(1+\varrho)^{\frac{1}{2}}-2a(1+\varrho)^{-a}}{(1+\varrho)^{\frac{1}{2}}-1}$:† consequently the total present value of the annuity is equal to $\frac{1}{2a}\times\frac{a-h(1+\varrho)^{\frac{1}{2}}}{(1+\varrho)^{\frac{1}{2}}-1}$.

In like manner, the present value of a life annuity payable *Quarterly* will, upon the same hypothesis, be equal to

$$\frac{1}{4a}\Bigg[\frac{a-\frac{1}{4}}{(1+\varrho)^{\frac{1}{4}}}+\frac{a-\frac{1}{2}}{(1+\varrho)^{\frac{1}{2}}}+\frac{a-\frac{3}{4}}{(1+\varrho)^{\frac{3}{4}}}+\frac{a-1}{(1+\varrho)}+\ldots\ldots\frac{a-a}{(1+\varrho)^{a}}\Bigg]$$

: which may be divided into the two following series

$$\frac{1}{4}\Bigg[\frac{1}{(1+\varrho)^{\frac{1}{4}}}+\frac{1}{(1+\varrho)^{\frac{1}{2}}}+\frac{1}{(1+\varrho)^{\frac{3}{4}}}+\frac{1}{(1+\varrho)}+\ldots\ldots\frac{1}{(1+\varrho)^{a}}\Bigg]-\frac{1}{16a}\Bigg[\frac{1}{(1+\varrho)^{\frac{1}{4}}}+\frac{2}{(1+\varrho)^{\frac{1}{2}}}+\frac{3}{(1+\varrho)^{\frac{3}{4}}}+\frac{4}{(1+\varrho)}+\ldots\ldots\frac{4a}{(1+\varrho)^{a}}\Bigg].$$

But, the first

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* See my *Doctrine of Interest and Annuities,* page 56.

† *Ibid.* page 89.

of these series is equal to $\frac{1}{4} \times \frac{1-(1+\varrho)^{-a}}{(1+\varrho)^{\frac{1}{4}}-1} = q$;* and the second is equal to $- \frac{1}{16a} \times \frac{4q(1+\varrho)^{\frac{1}{4}}-4a(1+\varrho)^{-a}}{(1+\varrho)^{\frac{1}{4}}-1}$: consequently the total present value of the annuity is in this case equal to $\frac{1}{4a} \times \frac{a-q(1+\varrho)^{\frac{1}{4}}}{(1+\varrho)^{\frac{1}{4}}-1}$.

By a similar process we might find the present value of such annuities payable at any other intervals: but it will be sufficient to show the extreme limit of the increase which arises from thus supposing the annuity to be payable at such smaller intervals: this limit takes place when the annuity is considered as being paid *momently*; in which case the expression becomes $\frac{a-m}{a \times \text{NL.}(1+\varrho)}$: where m is equal to $\frac{1-(1+\varrho)^{-a}}{\text{NL.}(1+\varrho)}$, or, to the present value of an annuity *certain* for the term a, payable *momently*.†

§ 355. If the numerical value of these expressions for half-yearly and quarterly annuities, according to the hypothesis of M. De Moivre, be compared with those deduced from real observations, they will be found to confirm the accuracy of each other; and to justify the rule which I have before given: namely, that the value of annuities payable *yearly* must be increased nearly

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* See my *Doctrine of Interest and Annuities*, page 56.

† *Ibid.* page 58. And I would here observe that I take NL to denote the *Neperean* logarithm of the quantity immediately following it.

$\frac{1}{4}$ of a year's purchase in order to show the value of the same annuities payable *half-yearly*; and that they must be increased nearly $\frac{3}{8}$ of a year's purchase in order to show the value of the same annuities payable *quarterly*; and also that they must be increased by $\frac{1}{2}$ a year's purchase in order to show the value of the same annuities payable *momently*.

§ 356. The reader must observe that, in all these cases, I have had regard only to the true rate of *annual* interest, agreeably to the principles which I have laid down in another work* for determining the value of annuities in general. But such annual rate may always be expressed in terms of the *nominal* rate, by making the substitutions there alluded to,† according as the interest is payable half-yearly, quarterly, &c: whereby we shall find that, on M. De Moivre's hypothesis, the present value of an annuity, on the life whose complement is a, payable *yearly, half-yearly, quarterly*, and *momently*, and on the supposition that the *interest* is also payable at the *same* periods, will be denoted respectively by $\frac{a-(1+r)y}{ar}$, $\frac{a-(1+\frac{r}{2})h}{ar}$, $\frac{a-(1+\frac{r}{4})q}{ar}$, and $\frac{a-m}{ar}$: where y, h, q, and m will *now* respectively denote the present values of annuities certain for the term a, payable *yearly*, *half-yearly*, *quarterly* and *momently*, and on the supposition that the *interest* also is payable at the *same* periods.

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* See my *Doctrine of Interest and Annuities*, page 55.

† That is, by substituting $\left(1+\frac{r}{m}\right)^{m}-1$ for ϱ: agreeably to what I have said in my *Doctrine of Interest and Annuities*, § 77, page 54.

If the numerical value of *these* expressions be compared with each other, it will be found that the half-yearly annuities will, upon this principle, be about $\frac{2}{10}$ of a year's purchase, and quarterly annuities about $\frac{3}{10}$ of a year's purchase, more than the value of the same annuities payable yearly; and this is the rule given by Dr. Price for such purpose.* But as the periods of the payments of the *annuity* are totally independent of the periods of the payment of *interest*, and ought not to be confounded together (as I have more fully explained in the tenth Chapter of my *Doctrine of Interest and Annuities*), we shall find that the addition to the tabular values of $\frac{1}{4}$ and $\frac{3}{8}$ respectively, as stated in pages 333 and 335, will be the most correct rule for general use: agreeably to what has been

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* See his *Observations on Rev. Pay.* vol. 1, page 246; and the investigations which are there annexed. The two following examples, given by him, will show the real difference which arises in these cases.

Value of an Annuity on a Single Life,
Interest 4 per cent.

Age.	Payable Yearly.	Payable Half-yearly.	Payable Quarterly.	Payable Momently.
36	13·829	14·010	14·101	14·191
61	8·753	8·973	9·072	9·199

already advanced by Mr. Simpson in his *Doctrine of Annuities and Reversions*, page 79.

As the two methods of investigation are however now before the public, the computist may adopt that rule which he conceives to be best suited to the circumstances of the case. Dr. Price had endeavoured to overturn Mr. Simpson's rule without stating the grounds of his dissent, and to substitute his own without explaining the nature and cause of their difference.

§ 357. Hitherto I have considered the differences in the value of annuities on *single* lives only: but it will be evident that, on the supposition that money is improved at a given *annual* rate of interest, the differences will be nearly the same on two *joint* lives, whose value is deduced from real observations, and whose probabilities of living to the end of every half-year are respectively denoted by $\frac{ab + a'b'}{2ab}$, $\frac{a'b' + a''b''}{2ab}$, $\frac{a''b'' + a'''b'''}{2ab}$, &c: which probabilities, though not strictly correct, may answer the present purpose.

On M. De Moivre's hypothesis, however, and on the supposition that money can be improved at interest payable at the *same* periods as the annuity, the value of an annuity on the two joint lives whose complements are a and b, payable *half-yearly*, is accurately expressed by the following series $\frac{1}{2ab}\left[\frac{(a-\frac{1}{2}).(b-\frac{1}{2})}{(1+\frac{r}{2})} +\right.$

$\frac{(a-1).(b-1)}{(1+\frac{r}{2})^2} + \frac{(a-\frac{3}{2}).(b-\frac{3}{2})}{(1+\frac{r}{2})^3} + \frac{(a-2).(b-2)}{(1+\frac{r}{2})^4} + \ldots\ldots \frac{(a-b).(b-b)}{(1+\frac{r}{2})^{2b}}\Big]$: the sum of which may, from what has preceded, be easily found equal to $\frac{1}{r} - \frac{2+r}{2ar}\Big[\frac{h}{b} \times \left(a-b-\frac{1}{2}-\frac{2}{r}\right) + \frac{2}{r}\Big]$.

In like manner the value of an annuity on those lives, payable *quarterly*, is on the same principles equal to $\frac{1}{4ab}\Big[\frac{(a-\frac{1}{4}).(b-\frac{1}{4})}{(1+\frac{r}{4})} + \frac{(a-\frac{1}{2}).(b-\frac{1}{2})}{(1+\frac{r}{4})^2} + \frac{(a-\frac{3}{4}).(b-\frac{3}{4})}{(1+\frac{r}{4})^3} + \frac{(a-1).(b-1)}{(1+\frac{r}{4})^4} + \ldots\ldots \frac{(a-b).(b-b)}{(1+\frac{r}{4})^{4b}}\Big]$ the sum of which is equal to $\frac{1}{r} - \frac{4+r}{4ar}\Big[\frac{q}{b}\left(a-b-\frac{1}{4}-\frac{2}{r}\right) + \frac{2}{r}\Big]$.

And, if the annuity and the interest are both supposed to be payable *momently*, its value will come out equal to $\frac{1}{r} - \frac{1}{ar}\Big[\frac{m}{b}\left(a-b-\frac{2}{r}\right) + \frac{2}{r}\Big]$. In which several formulæ, the quantities h, q and m denote the present value of annuities *certain* for the term b, payable *half-yearly*, *quarterly* and *momently*, on the supposition that the *interest* also is payable at the *same* periods respectively.*

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* The following examples, given by Mr. Morgan, will show the real difference that arises in these cases.

Value of an annuity on two Joint Lives,
Interest 4 per cent.

Ages.	Payable Yearly.	Payable Half-yearly.	Payable Quarterly	Payable Momently.
20—36	11·227	11·427	11·565	11·629
36—36	10·394	10·600	10·703	10·808
36—61	7·448	7·673	7·793	7·901
61—61	6·144	6·374	6.517	6·602

On Half-yearly &c Assurances.

§ 358. By a similar method of proceeding to that which has been adopted in the former part of this Chapter we might determine the present value of ASSURANCES for every *half*-year, *quarter*-year, &c of human existence. For, if the number of persons living at the end of $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, &c years from the age of A be denoted by the same quantities as in page 332, then will the probabilities of such life becoming extinct *in* the first, second, third, &c, *half*-years, be respectively represented by $\frac{a-\overset{\prime}{a}}{2a}$, $\frac{a-\overset{\prime}{a}}{2a}$, $\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{2a}$, $\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{2a}$, $\frac{\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a}}{2a}$, &c: consequently the present value of an assurance of the sum s on the life A for every *half*-year of human existence will be truly expressed by

$$\frac{s}{2a}\left[\frac{a-\overset{\prime}{a}}{(1+\varrho)^{\frac{1}{2}}}+\frac{a-\overset{\prime}{a}}{(1+\varrho)}+\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{(1+\varrho)^{\frac{3}{2}}}+\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{(1+\varrho)^{2}}+\frac{\overset{\prime\prime}{a}-\overset{\prime\prime\prime}{a}}{(1+\varrho)^{\frac{5}{2}}}+\&c\right]=\frac{s}{2}\left[1+(1+\varrho)^{\frac{1}{2}}\right]\times\frac{1-\varrho A}{(1+\varrho)}.$$

§ 359. In like manner, if the number of persons living at the end of $\frac{1}{4}$, $\frac{3}{4}$, $1\frac{1}{4}$, &c years from the age of A be denoted by the same quantities as in page 334, then will the probabilities of such life becoming extinct *in* the first, second, third, &c *quarter*-years, be respectively represented by $\frac{a-\overset{\prime}{a}}{4a}$, $\frac{a-\overset{\prime}{a}}{4a}$, $\frac{a-\overset{\prime}{a}}{4a}$, $\frac{a-\overset{\prime}{a}}{4a}$, $\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{4a}$, $\frac{\overset{\prime}{a}-\overset{\prime\prime}{a}}{4a}$, &c: consequently the present value of an assurance of the sum s on the life A for every *quar-*

ter-year of human existence will be truly expressed by

$$\frac{s}{4a}\left[\frac{a-a'}{(1+\varrho)^{\frac{1}{4}}}+\frac{a-a'}{(1+\varrho)^{\frac{1}{2}}}+\frac{a-a'}{(1+\varrho)^{\frac{3}{4}}}+\frac{a-a'}{(1+\varrho)}+\frac{a'-a''}{(1+\varrho)^{\frac{5}{4}}}+\frac{a'-a''}{(1+\varrho)^{\frac{6}{4}}}+\&c\right]=\frac{s}{4}\left[1+(1+\varrho)^{\frac{1}{4}}+(1+\varrho)^{\frac{1}{2}}+(1+\varrho)^{\frac{3}{4}}\right]\times\frac{1-\varrho A}{(1+\varrho)}=\frac{s}{4}\times\frac{\varrho}{(1+\varrho)^{\frac{1}{4}}-1}\times\frac{1-\varrho A}{(1+\varrho)}.$$

§ 360. By continuing these subdivisions we shall find that the present value of the same assurance on the life A for every n^{th} part of a year, will be truly expressed by $\frac{s}{n}\left[1+(1+\varrho)^{\frac{1}{n}}+(1+\varrho)^{\frac{2}{n}}+\ldots\ldots\right.$ $\left.(1+\varrho)^{\frac{n-1}{n}}\right]\times\frac{1-\varrho A}{(1+\varrho)}=\frac{s}{n}\times\frac{\varrho}{(1+\varrho)^{\frac{1}{n}}-1}\times\frac{1-\varrho A}{(1+\varrho)}$. Now, when n is infinite, this formula becomes equal to $\frac{s.\varrho}{\text{NL.}(1+\varrho)}\times\frac{1-\varrho A}{(1+\varrho)}$:* and which consequently denotes the value of the assurance for every *moment* of human existence; that is, the value of the given sum to be received *immediately* on the extinction of the given life.†

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* For, in such case $n\left[(1+\varrho)^{\frac{1}{n}}-1\right]$ is equal to the Neperean logarithm of $(1+\varrho)$. See Euler's *Introd. in Anal. Inf.* vol. 1, chap. 7, § 119; and also what has been said on this subject in my *Doctrine of Interest and Annuities*, page 46.

† Since the Neperean logarithm of $(1+\varrho)$ differs but little from $\frac{2\varrho}{2+\varrho}$, this last formula may be rendered more convenient for practice by means of the expression $\frac{s\varrho(2+\varrho)}{2\varrho}\times\frac{1-\varrho A}{(1+\varrho)}=s\left(1+\frac{\varrho}{2}\right)\times\frac{1-\varrho A}{(1+\varrho)}$: which exceeds the value of a yearly assurance, deduced from the rule in page 146, by the quantity $\frac{s\varrho}{2}\times\frac{1-\varrho A}{(1+\varrho)}$

It may be necessary to remark that these values are all deduced from the true *annual* rate of interest; which may be reduced to the *nominal* rate by making the substitution alluded to in the note in page 338.

On Life Annuities secured by Land.

§ 361. A life annuity *secured by land** differs from that kind of life annuity which has been treated of in the preceding part of this work, inasmuch that if the annuitant dies at any time between the stated periods for the payment of the annuity, his heirs are to receive such a sum as will be proportional to the time elapsed between the last payment and his death; whereas in all the cases hitherto considered, if the annuitant dies on the day preceding the time of payment, or sooner, his heirs cannot claim any part or portion of the annuity.

In this case, supposing the annuity payable yearly, the annuitant (since there is the same chance for his dying in one half of any year as in the other) may be considered as having an expectation of half a year's payment more than he would be otherwise entitled to. But the value of the half of £1 to be received on the extinction of any life, A, is by Prob. XXII equal to $\frac{1-\varrho A}{2(1+\varrho)}$; and, which is the addition that ought to be made to the value of an annuity payable yearly, in order to obtain its value when secured by land: consequently the value of such annuity is $A + \frac{1-\varrho A}{2(1+\varrho)}$ $= \frac{1+(2+\varrho)A}{2(1+\varrho)}$.

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* See on this subject, Dodson, vol. iii. Ques. 1 to 4, and 8 to 14. Price, vol. 1, page 244.

§ 362. In like manner, supposing the annuity payable half-yearly, the annuitant may be considered as having an expectation of a quarter of a year's payment more than he would be otherwise entitled to. But the value of the quarter of £1 to be received on the extinction of the life A in any half year, is, by the formula in page 342, equal to $\frac{\left[1+(1+\varrho)^{\frac{1}{2}}\right] \times \left[1-\varrho A\right]}{8(1+\varrho)}$; and which is the addition that ought to be made to the value of an annuity payable half-yearly, in order to obtain its value when secured by land. And so on for the additions that ought to be made to the value of an annuity payable quarterly &c. But the *difference* between the value of an annuity payable yearly, *not* secured by land, and the value of an annuity, payable at the same, or at any other intervals, which *is* secured by land, can in no case exceed ·05, or half unity.

§ 363. M. De Moivre in his *Doctrine of Chances*, page 338, has given a theorem for finding the value of an annuity *secured by land* and payable yearly, which he deduced by a differential process; a method easily applicable to his hypothesis: and Mr. Dodson in the third volume of his *Mathematical Repository*, page 4, has given another theorem for that purpose (obtained without the aid of that calculus) which brings out nearly the same answers.* But Mr. Simpson in his *Select Exercises*, page 323, and in the *Supplement* to his Doctrine of Annuities, page 70, has given a theorem which shows the value, not of an annuity payable yearly and secured by land, but of an annuity payable *momently* at a given *annual* rate of interest.† The values in all these cases being obtained from M. De Moivre's hypothesis.

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* M. De Moivre's formula is $\frac{1}{\varrho} - \frac{y}{a \times \text{NL}.(1+\varrho)}$: where y denotes the value of an annuity *certain* for the term a, payable yearly. Now the Neperean logarithm of $(1+\varrho)$ is very nearly equal to $\frac{2\varrho}{2+\varrho}$: if, therefore, we substitute this latter quantity, instead of NL.$(1+\varrho)$, the above formula will become $\frac{a-y\left(1+\frac{\varrho}{2}\right)}{a\varrho}$; which is the same as that given by Mr. Dodson, and which exceeds the value of an annuity not secured by land (as deduced in page 316) by the quantity $\frac{y}{2a}$.

† Mr. Simpson's formula is $\frac{1}{\text{NL}.(1+\varrho)} - \frac{m}{a \times \text{NL}.(1+\varrho)} = \frac{a-m}{a \times \text{NL}.(1+\varrho)}$; which is the same as that given in § 354, for determining the value of a life annuity payable *momently*, at a given *annual* rate of interest: but this is certainly not a correct mode of proceeding in order to find the value of an annuity *secured by land*.

Dr. Price is wrong in asserting that "Mr. Simpson makes the "excess of the value of such an annuity, above the value of an "annuity payable yearly but not secured by land, *double* to the "same excess derived from Mr. Dodson's and M. De Moivre's "rules." The truth is, that not only Dr. Price, but Mr. Simpson himself, appear to have been deceived by the similarity of the symbols employed in the two formulæ compared; without sufficiently considering that these symbols denote *different* quantities.

I would here observe that the formulæ, which I have given above, are the first that have been deduced from real observations; and are much more simple than those deduced from M. De Moivre's hypothesis. But though they readily follow, after the investigations that have been previously entered into; and might easily have been adapted, by preceding writers, to the value of annuities as deduced from such observations; yet those, who have been the most forward to attack the whole of M. De Moivre's principles, have not only suffered his formulæ on this, *and on other subjects*, to remain uncorrected and unreproved; but have inserted them in their works as affording a *proper* and *correct* solution to such cases!!!

CHAPTER XI.

On the Value of Deferred Annuities, Reversionary Annuities, and Assurances, in Annual Payments.

§ 364. In all those cases of deferred annuities mentioned in Prob. I, cor. 3, and in the corollaries to the subsequent problems, as well as in all cases of Assurances, I have deduced the values of the same in one *single* payment: but it is oftentimes required to determine such values in *annual* payments. The method of doing which I shall now proceed to show.

In the case of *Deferred* annuities depending on any number of joint lives ABC, the value in one single payment is (by Prob. I, cor. 3) denoted by $(ABC)^{\text{d}}$. Now, if the purchaser of this annuity is desirous of paying for the same by equal *annual* payments during the given term,* those equal payments ought to be such that their total present value shall be equal to the single payment above mentioned: or, in other words, he should pay instead of such sum an *equivalent annuity* during the given term.

§ 365. Let the required annual payment be denoted by p; and let the value of a temporary annuity on the given lives (that is, of an annuity to continue till the period when the deferred annuity commences) be denoted by $(ABC)^{\mathrm{t}}$: then, since the value of the deferred annuity, or $(ABC)^{\mathrm{d}}$, is to be paid for by equal annual payments during the time such annuity is deferred (subject to failure if any of the given lives become extinct in that period), it is evident that the sum or value of such payments must be equal to the value of an annuity, on the given lives *for such time*, of the yearly value of p: that is, $p\,(ABC)^{\mathrm{t}} = (ABC)^{\mathrm{d}}$.

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* Such annual payments, however, subject to failure if the given lives become extinct before the end of that period.

This, however, is on the supposition that the first annual payment is not made till the *end* of the first year, and continued at the *end* of every subsequent year till the expiration of the term. But this rarely, if ever, happens: and the usual, if not the invariable method is to advance the first payment *immediately;* and the remaining ones at the *beginning* of each of the following years: so that the number of payments shall be equal to the number of years during which the annuity is deferred. Therefore (since the payment which was supposed, in the preceding case, to be made at the *end* of the *term* is now made at the *beginning*) we must add unity to the value of a temporary annuity for *one year less than** the given term: and this quantity multiplied by the annual payment will be equal to the value of the deferred annuity. Consequently the formula will, in this case, become $p\left[1+(ABC)^{t-1}\right] = (ABC)^{d}$; whence, $p = \frac{(ABC)^{d}}{1+(ABC)^{t-1}}$:* and whence the following rule.

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* Dr. Price, in *all* the cases of annual payments which he has given, says that we must add unity to the value of a temporary annuity *for the given term:* by which means he makes the number of payments to be one more than ever occurs. The reader should particularly observe this in comparing *his* rules with the formula here given.

§ 366. *Divide the value of the Deferred annuity, by unity added to the value of a similar Temporary annuity for one year less than the given term: the quotient will be the annual payment required.*

For examples of the use and application of this rule, see the Scholium to Question VI in Chapter XII.

§ 367. The same rule will apply to the case of deferred annuities depending on the *longest* of two or more lives: see Prob. II, cor. 2. For, if the value of an annuity on the longest of any number of lives be denoted by L, then will the value of a similar *deferred* annuity be denoted by $(L)^{d}$; and also the value of a similar *temporary* annuity for *one year less* than the given term will be denoted by $(L)^{t-1}$. Consequently, from what has been above said, we shall have $p = \frac{(L)^{d}}{1+(L)^{t-1}}$.

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* Since $(ABC)^{t-1} = (ABC)^{t} - \frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n} = ABC - (ABC)^{d} - \frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n}$; it follows that the formula given in the text may be denoted by $\frac{(ABC)^{d}}{1+ABC-\left[(ABC)^{d}+\frac{\alpha\beta\gamma}{abc}(1+\varrho)^{-n}\right]}$, which will be oftentimes found very convenient in practice.

For examples of the use and application of this formula, see Question XI in Chapter XII.

In this case, however, it should be particularly observed that if the deferred annuity depends on the *joint* continuance of the given lives to the end of the given term (as mentioned in Prob. II, cor. 3) the formula will become $p = \frac{(L)^{d}}{1+(ABC)^{t-1}}$.

For examples of the use and application of this formula, see the Scholium to Question XI in Chapter XII.

§ 368. A similar method of reasoning will lead us to the true value, in annual payments during the continuance of the given lives, of any *Reversionary* annuity. Thus, let the value of the reversionary annuity, mentioned in the first case in page 71, be denoted by R; then will the value of the same in annual payments, during the joint continuance of the two lives, be $p = \frac{R}{1+AP}$. The same formula extends also to the case of *Deferred* reversionary annuities.

For examples of the use and application of this formula, see the Scholium to Question XIII, in Chapter XII: and also Question XVIII, and the Scholium to Question XVIII in that chapter.

But, if the reversionary annuity be *Temporary*, or for a given term only, and such annuity be denoted by $(R)^{t}$, we shall have $p = \frac{(R)^{t}}{1+(AP)^{t-1}}$.

For examples of the use and application of this formula, see Question XIX in Chapter XII.

§ 369. The principles here laid down will likewise extend to all the cases of *Assurances* mentioned in Chapter VI; whether for the whole continuance of life, or for any given term. For, if the present value of an assurance of any given sum be denoted by S, and the present value of a *temporary* assurance of a similar sum be denoted by $(S)^{t}$; then will the equivalent annuity during the joint continuance of all the lives involved be, in the first case, $p = \frac{S}{1+ABC}$; and in the latter case $p = \frac{(S)^{t}}{1+(ABC)^{t-1}}$.

It is scarcely necessary to observe, when S denotes the value of an assurance on the *longest* of any number of lives, that ABC will in such case denote the value of an annuity on the *longest* of such lives: agreeably to what has been said in Prob. XXII, cor. 2. And so likewise of any other assurance there alluded to.

For the use and application of the formula, see Questions XXVI, XXVII, and XXIX in Chapter XII.

§ 370. With respect to those assurances which are the subject of Chapter VIII, the annual payment may be divided into three kinds: 1° where such payment is made till the *claim is determined:* 2° where it is made till the *sum becomes due:* 3° where the sum becomes due *at the time* the claim is determined.

Thus in Prob. XXVII, the sum becoming due at the same time that the claim is determined, the value of the annual payment is obtained by dividing the value of the assurance by unity added to the value of an annuity on the two joint lives AB: that is $p = \frac{S}{1+AB}$.

In Prob. XXVIII, the *claim is determined* on the extinction of the two joint lives: but the sum does not *become due** till the extinction of A's life. Therefore the value of the annual payment till the *claim is determined* will be $p = \frac{S}{1+AB}$: and the value of the annual payment till the sum *becomes due* is $p = \frac{S}{1+A}$.

In Problem XXIX the sum becomes due at the time the claim is determined; and consequently the annual payment is equal to the value of the assurance divided by unity added to the value of an annuity on the three joint lives ABC: that is $p = \frac{S}{1+ABC}$.

In Prob. XXX the claim is not determined, neither does the sum become due, till the extinction of the joint lives AB, and also of the joint lives AC. That is, the annual payment must be made during the continuance of the joint lives AB, and likewise during the continuance of the joint lives AC after the decease of B. The two values will be found equal to $AB + AC - ABC$: consequently we shall have in this case $p = \frac{S}{1+AB+AC-ABC}$.

In Prob. XXXI the annual payment, till the *claim*

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* That is, provided the claim is determined *in favour* of the person assuring: and this must be understood in all these cases.

is determined, will be the same as in the last problem: but the value of such payment till the *sum becomes due* is evidently $p = \frac{S}{1+A}$.

In a similar manner we might proceed with respect to the remaining problems in Chapter VIII: but enough has here been said to enable the reader to determine the annual payment in any other case, either of annuities or assurances, that may arise in practice: I therefore shall not detain him with any further remarks on this subject.

CHAPTER XII.

Practical Questions to illustrate the Use of some of the preceding Problems.

QUESTION I.

§ 371. To find the *probability* that a life or lives, of any given age, will continue in being to the end of any given term; according to any given table of observations.

SOLUTION.

In the case of a *single* life, this probability is a fraction whose denominator is the number of persons living at the given age; and whose numerator is the number of persons living at an age older by the given term than the given age.

In the case of *joint* lives it is the product of the probabilities that each of the single lives shall continue in being to the end of the given term. See § 23.

Example 1. The probability that a person, whose age is 20, shall attain to the age of 50, or live 30 years, is, according to the observations of *M. De Parcieux* as given in Table III, equal to $\frac{581}{814}$. And the probability that a person, whose age is 40, shall attain to the age of 70, or live 30 years, is, according to the same observations, equal to $\frac{310}{657}$.

But the probability that *both* those persons shall live to the end of 30 years is equal to $\frac{581}{814}$ multiplied by $\frac{310}{657}$: that is, equal to $\frac{180110}{534798}$.

Example 2. The probability that a man aged 46 shall attain to the age of 56, or live 10 years, is, according to the observations made in *Sweden,* as given in Table XIV, equal to $\frac{3096}{3991}$. And the probability that a woman aged 40 shall attain to the age of 50, or live 10 years, is, according to the same observations, equal to $\frac{4027}{4733}$.

But the probability that *both* those persons shall live 10 years is equal to $\frac{3096}{3991}$ multiplied by $\frac{4027}{4733}$: that is, equal to $\frac{12467592}{18889403}$.

Example 3. The probability that *each* of three lives aged 20, 30, and 40 shall live 15 years, is, according to the observations made at *Northampton*, as given in Table XXV, equal to $\frac{4010}{5132}$, $\frac{3248}{4385}$, and $\frac{2448}{3635}$ respectively.

But the probability that *all* those lives shall continue so long is equal to the product of the three fractions into each other: whence, such probability will be denoted by $\frac{31883927040}{81801385700}$.

SCHOLIUM.

§ 372. Having thus found the probability that any single or joint lives will continue in being to the end of any given term, we may readily determine the probability that *one or the other* of them will live so long. For, in the case of two lives, the probability here alluded to will be equal to the *difference* between the probability that the joint lives will continue to the end of the term and the sum of the probabilities that each of the single lives will continue so long.

Thus, in the first example, the probability that one or other of two lives, aged 20 and 40, will continue 30 years is equal to $\frac{180110}{534798}$ subtracted from $\frac{634057}{534798}$ (or from the sum of the two quantities $\frac{581}{814}$ and $\frac{310}{657}$):* which leaves $\frac{453947}{534798}$ for the probability required.

And the probability that one or other of the two lives, mentioned in the second example, will continue 10 years, is equal to $\frac{12467592}{18889403}$ subtracted from $\frac{30725125}{18889403}$ (or from the sum of the two quantities $\frac{3096}{3991}$ and $\frac{4027}{4733}$): which leaves $\frac{18257533}{18889403}$ for the probability required.

In like manner the probability that some one or other, out of three given lives, will continue to the end of any given term, is found by subtracting the sum of the probabilities that each pair of joint lives will continue so long, from the sum of the probabilities that each single life and that the three joint lives will continue the given time: agreeably to the principles laid down in Prob. II.*

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* These fractions, reduced to a common denominator, are $\frac{381717}{534798}$ and $\frac{252340}{534798}$, the sum of which is equal to $\frac{634057}{534798}$. But it is tedious to operate in this way, and I have adopted it in the present instance for the sake of illustration only. The best method of finding the probabilities, both for single and joint lives, is by means of logarithms: and I would here observe that the logarithm of the denominator subtracted from the logarithm of the numerator will give the logarithm of the probability required: which logarithm will always have a *negative index*.

QUESTION II.

§ 373. To find the *expectation* of any given life (or lives) receiving a given sum, at the end of any given term.

SOLUTION.

Multiply the present value of the given sum by the probability that the given life (or lives) will continue to the end of the given term: the product will be the answer required. See the note in page 44.

Example 1. What is the present value of £1 to be received at the end of 30 years, provided a person, now aged 20, be then alive: interest being reckoned at 4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

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* I shall here mention, by way of note, that the probability that any *two out of three* given lives will continue to the end of any given term, is equal to twice the probability that the three joint lives shall continue the given time, subtracted from the sum of the probabilities that each pair of joint lives shall continue the same period: agreeably to what has been said in the investigation of Prob. III.

The present value of £1 to be received at the end of 30 years, without any contingency, is by Table LVIII equal to ·26700; and the probability that a person aged 20 will live 30 years is, by the preceding Question, equal to $\frac{581}{814}$: therefore, these two quantities multiplied together* will produce ·1906 for the value required.

In like manner the expectation of receiving that sum at the end of the same period, provided a person, aged 40, lives so long, is equal to $\frac{310}{657}$ multiplied by ·26700; which produces ·12598 for the value in this case required.

But, if the expectation depended on *both* those lives continuing to the end of the term; then $\frac{180110}{534798}$, multiplied by ·26700, will produce ·08992 for the value required.

And, if it had depended on *either* of those lives continuing to the end of the term, then $\frac{453947}{534798}$ (or the value found by the Scholium in page 357) being multiplied by ·26700 will produce ·22663 for the value of the expectation in such case required.

Example 2. A man aged 46 will at the expiration of a lease, which has 10 years to run, be entitled to a fine of £1,† provided he be then alive: what is his expectation of receiving the same: interest being reckoned at 4 per cent, and the probabilities of living as observed in *Sweden?*

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* The method of multiplying a vulgar fraction by a decimal fraction is, to multiply the *decimal* by the *numerator* of the vulgar fraction, and to divide the product by the *denominator* of the same.

† I have taken the fine equal to *one* pound, because the quantities which result from this assumption will be often referred to in the course of the present chapter: but, it is easy to see that the answer here obtained, being multiplied by *any other* fine,

The present value of £1 certain to be received 10 years hence is, by Table LVIII, equal to ·67556; and the probability that a man aged 46 will live 10 years is, by the preceding question, equal to $\frac{3096}{3991}$: therefore these two quantities multiplied together will produce ·52406 for the value required.

Had the fine depended on the life of his wife aged 40, then $\frac{4027}{4733}$ multiplied by ·67556, will produce ·57479 for the value in this case required.

But had it depended on their *joint* lives continuing to the end of the given term, then $\frac{12467502}{18889403}$ multiplied by ·67556, will produce ·44589 for the value in such case required.

And had it depended on *either* of those lives continuing so long, then $\frac{18257533}{18889403}$ multiplied by ·67556 will produce ·65296 for the value in this case required.

SCHOLIUM.

§ 374. By means of the general solution here given may be determined all questions relative to the value of such sums as ought to be given for the

would give the present value of such other fine. Thus if the fine were £100, the present value of the same, if depending on the life of the man, would be equal to 52·406 or £52 : 8 : 1; and, if depending on the life of the woman, would be equal to 57·479 or £57 : 9 : 7.

Endowments of Children. Thus, suppose a person has a son aged 11, for whom he wishes to secure £100 on his coming of age; the sum which he ought to pay down for the assurance of the same (reckoning interest at 5 per cent, and the probabilities of living as according to *M. De Parcieux*) is equal to $\frac{806}{872}$ multiplied by 61·391; which produces 56·744, or £56 : 14 : 10 for the answer required.

In the table of rates published by the *Globe Assurance* Company, and by the *Provident Institution**, the sums demanded for the Endowments of Children are in general full as much as (and from the age of 9 years and upwards are even *more* than) the present values, at 5 per cent, of £100 *certain* to be received at the end of the given term *without any contingency*. For instance, £62 : 11 : 2 is required in ready money by the *Globe*, and £62 by the *Provident*, for the payment of £100 on the event of a child, aged 11, arriving at the age of 21 years: whereas either of these sums put out to interest at 5 per cent would amount to *more* than £100 at the end of that period, without the liability to loss in case the child should happen to die before that time!!! No person can, I think, be so blind to his own interest as to risk his money in this absurd way.

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* The other Companies have not published *their* rates for the Endowments of Children: but, from the similarity of most of the rates at the various Offices, we may conclude that there is no great difference on this point.

QUESTION III.

§ 375. To find the value* of an annuity on any *Single* life.

SOLUTION.

This value is determined by inspection: for, in either of the Tables which show the values of annuities on any single life, we shall find the value required set down against the age of the given life, according to the several rates of interest at the top of each column.

Example 1. The value of an annuity on a life aged 20, reckoning interest at $4\frac{1}{2}$ per cent, and the probabilities of living as observed by *M. De Parcieux*, is, by Table V, equal to 16·624; or about $16\frac{5}{8}$ years purchase.†

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[*] By the *value* of an annuity I mean the *number of years purchase* that such annuity is worth: agreeably to what I have already observed in the note in page 27: and as this mode of expression is used in all the subsequent questions, it will be necessary to bear this observation in mind.

[†] The *number of years purchase* being multiplied by the annuity will give the total present value of the same. Thus, if the annuity in the present instance were £50 per annum: then 16·624 multiplied by 50 would give 831·200, or £831 : 4 : 0 for the value of the same. Or, if the annuity had been £4 : 10 : 0 per annum; then 16·624 multiplied by 4·5 would give 74·808 or £74 : 16 : 2 for the value in this case required. This method is universal, and applies to all cases of annuities, whether present or in reversion; whether temporary or deferred: and therefore it will be sufficient, in all the subsequent examples, to deduce the value of an annuity

Had the life been 40 years of age, the value would have been equal to 14·254. Or had the rate of interest been in each case 5 per cent, the values would have been equal to 15·469 and 13·459 respectively.

Example 2. The value of an annuity on the life of a *man* aged 46, reckoning interest at 4 per cent and the probabilities of living as observed in *Sweden* is, by Table XVI, equal to 12·297; or rather more than 12¼ years purchase.

Had the annuity been on the life of a *woman* aged 40, the value would have been equal to 14·401. Or had the rate of interest in each case been 5 per cent, the values would have been 11·153 and 12·856 respectively.

QUESTION IV.

§ 376. To find the value of an annuity on *two Joint lives.*

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of *one pound* per annum: or, in other words, to find the *number of years purchase.*

Having thus found the number of years purchase that ought to be given for an annuity, we may readily determine the *annuity* that ought to be given for any given *sum* invested, merely by dividing such sum by the number of years purchase. Thus, if a person wished to lay out £4000 in the purchase of such an annuity as the one mentioned in the text, the annuity which he ought to receive for that money will be found by dividing 4000 by 16·624: whence 240·616, or £240 : 12 : 4 will be the annuity required. This method is likewise universal, and therefore it will be unnecessary to repeat it in any of the subsequent cases. The same principles will apply to the value of reversionary *sums;* for which, see Question XXVII.

SOLUTION.

Look in the Tables which show the values of annuities on two joint lives of all ages; and if the two lives have the same common age, or if their difference of age comes within the limits of those tables, the value of an annuity on their joint continuance will be found expressed therein.

Example 1. The value of an annuity on two joint lives aged 20 and 40, interest being reckoned at $4\frac{1}{2}$ per cent, and the probabilities of living as observed by *M. De Parcieux*, is, by Table IX, equal to 12·545; or rather more than $12\frac{1}{2}$ years purchase.

Had both the lives been 20 years of age, the value would have been, by Table VI, equal to 14·004: or, had they been both 40 years of age, the value would have been 11·710.

Example 2. The value of an annuity on the joint lives of a man aged 46 and his wife aged 40, reckoning interest at 4 per cent and the probabilities of living as observed in *Sweden*, is, by Table XVIII, equal to 10·286.

Had both the lives been 40 years old the value would have been, by Table XVII, equal to 10·964: or, had they been both 46 years old, the value would have been 9·736.*

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* The values of annuities on the joint lives in Tables XVII to XXIV are deduced from the probabilities of living amongst males and females *collectively:* and therefore do not show the *true* values of annuities on two joint lives, one of which is a *male* and the other a *female*. Tables formed upon this latter principle are still a *desideratum*. See the example in page 33.

SCHOLIUM.

§ 377. If the *difference* of age between the two lives is any number of years not given in the tables, the required value may be easily obtained by means of the following rule.

Find, by the tables, the value of an annuity on two joint lives whose difference of age is *greater* than, but at the same time *nearest* to, the difference of age between the proposed lives; and the oldest of which is of the same age with the oldest of the proposed lives. Find also, by the same tables, the value of an annuity on two joint lives whose difference of age is the *next less* to that just mentioned; and the oldest of which is, in like manner, of the same age with the oldest of the proposed lives. Then will the 1st, 2nd, 3rd, &c, arithmetical *mean** between the least and the greatest of these two values be the value required; according as one of the proposed lives is 1, 2, 3, &c years younger than the other.

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* The tables for the values of annuities on two joint lives, according to the *Northampton* observations, are the only ones where the difference of age is so small as *five* years. In the tables deduced from the observations in *Sweden*, the difference of age is *six* years: and in those deduced from the observations of *M. De Parcieux*, the difference of age is *ten* years. Consequently the 1st, 2nd, 3rd, &c arithmetical mean between the least and greatest of any two values, according to the *Northampton* tables, will be equal to the least value increased by 1, 2, 3, &c *fifths* of their difference: but according to the *Swedish* tables, it will be equal to the least value increased by 1, 2, 3, &c *sixths* of their difference: and according to the tables of *M. De Parcieux*, it will be equal to the least value increased by 1, 2, 3, &c *tenths* of their difference.

Example 1. Let it be required to find the value of an annuity on two joint lives aged 32 and 50; at the rate of 4½ per cent interest, and according to the probabilities of life as observed by *M. De Parcieux?*

That difference of age which is *greater* than the difference between these lives, but at the same time *nearest* to it, is 20: and the value of an annuity on two joint lives whose difference of age is 20 years, and the oldest of which is of the same age with the oldest of the proposed lives (that is, the value of an annuity on two joint lives aged 30 and 50) is, by Table IX, equal to 10·611. And the value of an annuity on two joint lives whose difference of age is next less to 20 (that is, whose difference of age is 10 years) and the oldest of which is of the same age with the oldest of the proposed lives (that is, the value of an annuity on two joint lives aged 40 and 50) is, by Table VIII, equal to 10·274. Therefore, these being the values of an annuity on two joint lives aged 30 and 50, and on two joint lives aged 40 and 50; it is evident that the value of an annuity on two joint lives, aged 32 and 50, will be nearly equal to the least of these two values increased by 8 *tenths* of the difference between them: or (which is the same thing) equal to the greatest value diminished by 2 tenths of their difference. Now, the difference between these values is equal to ·337; *one* tenth of which is equal to ·0337, and *two* tenths are therefore equal to ·067.

Consequently 10·611, diminished by ·067, will leave 10·544 for the value required of an annuity on the two joint lives aged 32 and 50.

Example 2. Let it be required to find the value of an annuity on two joint lives aged 20 and 60; at the rate of 4 per cent interest, and according to the probabilities of living as observed in *Sweden?*

The difference of age which is greater than the difference between these lives, but at the same time nearest to it, is 42: and the value of an annuity on two joint lives whose difference of age is 42 years, and the oldest of which is of the same age with the oldest of the proposed lives (that is, the value of an annuity on two joint lives aged 18 and 60) is, by Table XXIV, equal to 8·208. And the value of an annuity on two joint lives whose difference of age is 6 years less than 40, and the oldest of which is likewise of the same age with the oldest of the proposed lives (that is, the value of an annuity on two joint lives aged 24 and 60) is, by Table XXIII, equal to 8·097. Therefore, these being the values of an annuity on two joint lives aged 18 and 60, and on two joint lives aged 24 and 60; it follows that the value of an annuity on the two joint lives 20 and 60 will be nearly equal to the least of these two values increased by 4 *sixths* of the difference between them. Now, their difference being equal to ·111, it follows that *one* sixth of such difference will be ·0185; and *four* sixths of such difference will be ·074: which being added to 8·097 will give 8·171 for the required value of an annuity on the two joint lives aged 20 and 60.

Example 3. What is the value of an annuity on two joint lives aged 26 and 60; reckoning interest at 5 per cent and the probabilities of life as observed at *Northampton?*

The difference of age which is greater than the difference between these two lives, but at the same time nearest to it, is 35: and the value of an annuity on two joint lives whose difference of age is 35, and the oldest of which is equal to the oldest of the proposed lives (that is, the value of an annuity on two joint lives aged 25 and 60) is by Table XXXV equal to 7·383. And the value of an annuity on two joint lives whose difference of age is 5 years less than 35, and the oldest of which is also of the same age with the oldest of the proposed lives (that is, the value of an annuity on two joint lives aged 30 and 60) is, by Table XXXIV, equal to 7·292. Therefore, these being the values of an annuity on two joint lives aged 25 and 60, and on two joint lives aged 30 and 60; it follows that the value of an annuity on the two joint lives 26 and 60 will be nearly equal to the least of these values increased by 4 *fifths* of the difference between them: or nearly equal to the greatest of these values decreased by 1 *fifth* of their difference. Now, this difference being ·091, it is evident that *one* fifth of it is equal to ·018; which being deducted from 7·383,* will give 7·365 for the value required of an annuity on the two joint lives 26 and 60.

§ 378. Since the tables of the values of annuities on two joint lives according to the observations of *M. De Parcieux* are calculated only for such lives whose difference of age is *ten* years, it is evident that the method just laid down (for determining the values of annuities on two joint lives, whose difference of age is any intermediate number) will not be quite so correct as from those tables, calculated according to the observations of life in *Sweden*, where the difference of age is *six* years. Neither will these latter ones show the value, for such intermediate ages, so correctly as the tables calculated according to the observations of life at *Northampton*, where the difference of age is *five* years. In neither case will the error be very considerable: but in the latter case particularly (where the tables show the values of annuities on two joint lives of all ages whose difference is not more than 5 years) the error is so trifling as to be not worth considering. This will evidently appear from the following comparison (given by Dr. Price in his *Observations on Reversionary Payments*, vol. 2, page 359,) of the values of annuities on two joint lives of the ages therein mentioned, deduced from the *Northampton* observations, interest at 3 per cent.

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* Or we may add four fifths to the least value, which would give the same result.

Ages.	Value by Rule.	Correct Value.
18—14	14·972	14·978
18—15	14·858	14·864
18—16	14·744	14·744
18—17	14·630	14·626
45—31	10·862	10·869
45—32	10·802	10·811

Ages.	Value by Rule.	Correct Value.
45—33	10·742	10·751
45—34	10·682	10·688
66—27	7·092	7·095
66—28	7·076	7·080
66—29	7·060	7·063
66—30	7·044	7·046

In the higher rates of interest the agreement is greater.

Dr. Price was enabled to make this comparison by the Tables in the office of the *Equitable Society*; where, in order to lay the foundation of accuracy in conducting the business of the office, it has been thought necessary to compute minutely to four places of decimals the values by the *Northampton* observations, at 3 per cent, of two joint lives *for every possible difference of age*.

§ 379. When one of the given lives is under 10 years of age, we ought, in deducing the values agreeably to this rule, to attend particularly to the order of the difference between the values taken from the Tables: that is, to observe whether such difference is *in*creasing or *de*creasing. For instance, suppose it is required to determine the value of an annuity on two joint lives aged 9 and 30, interest at 3 per cent, and the probabilities of living as at *Northampton:* the rule directs us to find the value of an annuity on two joint lives aged 5 and 30, and on two joint lives 10 and 30, which are respectively equal to 13·762 and 14·150;

and that ·078 (or one fifth of the difference between them) being subtracted from the latter value, will give 14·072 for the value of an annuity on two joint lives aged 9 and 30. But the following comparison will show this to be incorrect: for if we take out the values of annuities on the several joint lives as under, viz.

5—30 = 13·762
10—30 = 14·150
15—30 = 13·734
20—30 = 13·286
25—30 = 12·966
30—30 = 12·589

it will be seen that (beginning at the bottom) the values gradually *in*crease till we come to the age of 10 and 30; and therefore that the value of an annuity on any two joint lives, one of which is 30 years of age and the other of any age between 10 and 30, will be deduced accurately enough by means of the rule above given. And this also would be the case with respect to the value of annuities on any two joint lives, one of which is 30 years and the other of any age *below* 10 years of age, provided the *de*crease commenced exactly at the age of 10 years: but it is probable that the decrease does not begin to take place till about the joint ages of 8 and 30;* and consequently that the value of an annuity on two joint lives aged 9 and 30 is *greater* than 14·150, instead of being *less*. The proper method therefore of finding the value of an annuity on the two joint lives aged 9 and 30 will be to take ·083 (or one fifth of the difference between 14·150 and 13·734) and *add* it to 14·150; which will give 14·233 for the value of an annuity on the two joint lives aged 9 and 30. These cases have never yet been noticed by any preceding writer; although they frequently occur in practice.

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* The period, at which this decrease commences, varies according to the rate of interest and according to the difference between the ages of the two lives.

QUESTION V.

§ 380. To find the value of an annuity on *three Joint* lives.

SOLUTION.

Look in Tables XLIII and XLIV; and if the three lives have the same common age, or if their difference of ages be 10 and 20 years, the value of an annuity on their joint continuance will be found expressed therein.

Example. The value of an annuity on three joint lives aged 20, 30 and 40, reckoning interest at 4 per cent and the probabilities of living as at *Northampton*, is equal to 8·986: but had all the lives been 20 years of age, the value would have been equal to 10·342; or had they all been 40 years of age, the value would have been equal to 7·865.

SCHOLIUM.

§ 381. It unfortunately happens that the two tables above mentioned are the only ones, that have been published, for determining the values of annuities on three joint lives. The labour of computing such tables is so very great, and the combinations of ages are so various, that it will probably be a long time before any person will undertake to finish what has been here begun: and till that is the case we may make use of the following general and very easy rule, given by Mr. Simpson, for finding the values of annuities on any *three*, from the values on any *two*, joint lives.

" Let A be the youngest, and C the oldest of the
" three proposed lives. Take the value of an annuity
" on the two joint lives B and C, and find the age of
" a single life D of the same value. Then find the
" value of an annuity on the two joint lives A and D,
" which will be the value required."

Example. What is the value of an annuity on three joint lives aged 10, 20, and 30; interest at 4 per cent, and the probabilities of living as at *Northampton?**

The value of an annuity on the two joint lives aged

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* It will readily appear that we can obtain the values of annuities on *three* joint lives more correctly from the *Northampton* tables of *two* joint lives, than from any other observations; because they are, as yet, the most comprehensive, and include the greatest variety of combined ages.

20 and 30 is, by Table XXX, equal to 11·873; which, being compared with the values in Table XXVII, will be found equal to the value of an annuity on a single life D aged $47\frac{17}{205}$,* or 47 years and 1 month. And the value of an annuity on the joint lives A and D (that is, on two joint lives aged 10 and $47\frac{17}{205}$) is, by the rule in the preceding scholium, equal to 10·474;† which is the value required.

Had the two oldest lives been both 40 years of age, and the youngest 20, the value of an annuity on the joint lives of the two former would, by Table XXVIII, be equal to 9·820; answering to a single life D aged $56\frac{157}{228}$. And the value of an annuity on the joint lives A and D (that is, on two joint lives aged 20 and $56\frac{157}{228}$) is, by the rule alluded to in the last note, equal to 8·601: which is the value required of an annuity on three joint lives aged 20, 40, and 40.

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* The value, in Table XXVII, which is next greater than 11·873, is 11·890; which is the value of an annuity on a single life aged 47. The difference between these values, or 17, is the *numerator* of the fraction: and the *denominator* is the difference between 11·685 (or the next less value to 11·873) and 11·890.

† The value of an annuity on two joint lives aged 10 and 47 is, by the rule in the preceding scholium, equal to 10·485: and the value of an annuity on two joint lives aged 10 and 48 is by the same rule equal to 10·356. The difference between these two values, or ·129, being multiplied by $\frac{17}{205}$ will give ·011; which being subtracted from 10·485 will leave 10·474 for the value required. This shows the true method of proceeding in such cases; but, if this fraction be either very small or does not differ much from unity, the error will not be considerable if (for the sake of more expedition) D is always taken for that age, whether greater or less, which answers most nearly to the value of the annuity on the joint lives B and C, without regarding the fraction.

Or, had the two youngest lives been 20, and the eldest 40 years of age; then the value of an annuity on two joint lives aged 20 and 40 would, by Table XXXII, be equal to 10·924: answering to a single life D aged $51\frac{1}{2}\frac{3}{0}\frac{3}{8}$. And the value of an annuity on the two joint lives A and D (that is, on two joint lives aged 20 and $51\frac{1}{2}\frac{3}{0}\frac{3}{8}$) is, by the rule alluded to in the preceding note, equal to 9·406: which is the value required of an annuity on three joint lives aged 20, 20 and 40.

The following table (computed from the probabilities of life as observed at *Northampton*, and reckoning interest at 4 per cent) will show how nearly the rule, above explained, approximates to the true values as given in Tables XLIII and XLIV.

Ages.	Value by Rule.	Correct Value.
10—20—30	10·474	10·438
15—25—35	9·836	9·738
20—30—40	9·097	8·986
25—35—45	8·390	8·313
30—40—50	7·651	7·571
35—45—55	6·884	6·816
40—50—60	6·046	5·994
45—55—65	5·175	5·145
50—60—70	4·235	4·219
55—65—75	3·308	3·298
10—10—10	12·206	12·200
15—15—15	11·376	11·274
20—20—20	10·516	10·342

Ages.	Value by Rule.	Correct value.
25—25—25	9·937	9·796
30—30—30	9·351	9·221
35—35—35	8·703	8·585
40—40—40	7·983	7·865
45—45—45	7·243	7·126
50—50—50	6·433	6·317
55—55—55	5·637	5·550
60—60—60	4·817	4·755
65—65—65	3·936	3·914
70—70—70	3·010	2·995
75—75—75	2·118	2·119

From which it may be inferred that this rule will give the values of annuities on three joint lives generally within a *ninth* or a *tenth*, and sometimes within less than a *twentieth* part of a year's purchase. It may also be observed that when the oldest of the three ages does not exceed 75, and the youngest is not less than 10, the error falls on the side of *excess:* and, consequently, that if ·05 (or a twentieth part of a year's purchase) be deducted from the values by the rule, we shall obtain the true value, in some cases, almost exactly; and, in most cases, much more nearly.

QUESTION VI.

§ 382. To find the value of a *Deferred* annuity on any *single* or *joint* lives.*

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* This Question is of considerable utility in enabling us to determine the best means of providing Annuities for the *benefit of Old Age;* as will be more fully explained in the following Chapter.

SOLUTION.

Find the value of an annuity on a life, or joint lives, as many years older than the given life, or joint lives, as are equal to the term during which the annuity is deferred; find also the expectation of the given life, or joint lives, receiving £1 at the end of that term: the product of these two quantities will be the answer required. See § 45.

Example 1. A person aged 20 wants to purchase an annuity for what may happen to remain of his life after the term of 30 years: what is the present value of the same, reckoning interest at 4½ per cent, and the probabilities of life as observed by *M. De Parcieux?*

The value of an annuity on a life aged 50, is, by Table V, equal to 11·921; and the expectation of a life, aged 20, receiving £1 at the end of 30 years, is, by Question II, equal to ·1906: therefore 11·921 multiplied by ·1906 will produce 2·272 for the number of years purchase required.

Had the life been 40 years of age, the value would have been equal to 6·221 multiplied by ·1260; which would produce ·784 for the value required.

Example 2. A man now aged 46 will at the end of 10 years come into possession of an annuity on his own life: what is the present value of the same, reckoning interest at 4 per cent, and the probabilities of living as observed in *Sweden?*

The value of an annuity on a male aged 56 is, by Table XVI, equal to 9·717; and the expectation that a man aged 46 will receive £1 at the end of 10 years is, by Question II, equal to ·5241: therefore these two quantities being multiplied together will give 5·093 for the value required.

Had the annuitant been a female aged 40, then 12·049 multiplied by ·5748 would give 6·926 for the value in this case required.

Example 3. Two persons aged 20 and 40 wish to purchase an annuity for the remainder of their joint lives after 30 years: what ought they to give for the same, reckoning interest at 4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on two joint lives aged 50 and 70 is, by Table IX, equal to 5·517; and the expectation of two joint lives, aged 20 and 40, receiving £1 at the end of 30 years is, by Question II, equal to ·0899: therefore the product of these two quantities will give ·496 for the value required.

Example 4. A man aged 46 together with his wife aged 40 are entitled to an annuity on their joint lives, to commence at the end of 10 years: what is the value of their interest therein, taking the probabilities of life as observed in *Sweden*, and the rate of interest at 4 per cent.?

The value of an annuity on the joint lives of two persons, a man aged 56 and a woman aged 50, is, by Table XVIII, equal to 7·874; which being multiplied by ·4459 (or the value of the expectation of two joint lives, aged 46 and 40, receiving £1 at the end of 10 years as found by Question II) will produce 3·511 for the value required.

SCHOLIUM.

§ 383. If, instead of determining the value of a deferred annuity in a *single* payment, we wish to determine the value of the same in *annual* payments during the term for which the annuity is deferred;* the amount of those annual payments is readily obtained by means of the following rule.

Divide the value of the annuity in a *single* payment, by unity added to the value of a similar temporary annuity for one year less than the given term: the quotient will be the annual payment required. See § 366.

Example 1. A person aged 20 wants to purchase an annuity for what may happen to remain of his life after the term of 30 years: what sum ought he to give *annually* to the end of that term† in order to have the same assured to him; reckoning interest at 4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

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* The first of those annual payments to be made *immediately*, and the remaining ones at the *beginning* of every subsequent year: since this is the usual method of making such annual payments.

† Such annual payments, however, subject to failure, in case the given life becomes extinct before the end of that term.

The value of this deferred annuity in a *single* payment is, by the first example to the Question, equal to 2·272; and the value of a similar temporary annuity for 29 years is, by the rule in the following Question,* equal to 14·161: therefore 2·272 divided by 15·161 will give ·150 for the value of the annual payments during the term deferred.

In like manner we might determine the value in annual payments, of an annuity on the life of a woman for what may happen to remain of it after 10 years; reckoning interest at 4 per cent, and the probabilities of living as observed in *Sweden*.

For, the value of this deferred annuity in a *single* payment is, by the first example to the Question, equal to 6·926; and the value of a similar temporary annuity for 9 years is, by the rule just alluded to, equal to 6·900: therefore 6·926 divided by 7·900 will give ·877 for the value of the annual payments required.

Example 2. A man aged 46 and his wife aged 40 are entitled to an annuity on their joint lives, to commence at the end of 10 years, but are willing to surrender their interest in the same for an equivalent annuity (commencing immediately) during such term:

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* A more convenient method however of determining *such* temporary annuities is expressed by the following rule. To the value of the deferred annuity add the expectation that the given life or lives shall receive £1 at the end of the given term; subtract the sum from the value of an annuity on the given life or lives: the difference will be the value of the temporary annuity for *one year less than the given term*. See the note in page 350.

what ought that equivalent annuity to be, reckoning interest at 4 per cent, and the probabilities of living as observed in *Sweden?*

The value of the deferred annuity on the joint lives is, by the fourth example to the Question, equal to 3·511; and the value of a similar temporary annuity for 9 years is, by the following Question (or the rule in the preceding note) equal to 6·329; therefore 3·511 divided by 7·329, will give ·479 for the value of the annual payments during the term deferred.

QUESTION VII.

§ 384. To find the value of a *Temporary* annuity on any *single* or *joint* lives.*

SOLUTION.

From the value of an annuity on the given single or joint lives, deduct the value of an annuity on the same lives *deferred* during the given term: the remainder will be the value required. See § 47.

Example 1. A person aged 20 buys an annuity for 30 years, on condition that if he dies before the expiration of that term, the annuity shall cease: what ought he to give for the same, reckoning interest at

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* I call a *temporary* annuity, one that is to continue during a given *term* only: which term is *less* than that to which it is possible the life or lives may extend. See the note in page 45.

4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on a life aged 20 is, by Table V, equal to 16·624; and the value of an annuity on the same life, deferred for 30 years, is, by Question VI, equal to 2·272: consequently, this value, subtracted from the former, will leave 14·352 for the answer required.

Had the life been 40 years of age, then ·784 (or the value of an annuity on such life deferred for 30 years, as found by Question VI) deducted from 14·254, would leave 13·470 for the value in this case required.

Or, had these two persons (aged 20 and 40) purchased the annuity on their *joint* lives, then ·496 (or the value of an annuity on such joint lives deferred for 30 years, as found by Question VI) being deducted from 12·545, will leave 12·049 for the value in this case required.

Example 2. A man aged 46 is entitled to the rent of an estate for 10 years, provided he lives so long: what is the value of his interest therein, reckoning interest at 4 per cent, and the probabilities of living as observed in *Sweden?*

The value of an annuity on such life is, by Table XVI, equal to 12·297; and the value of an annuity on the same life, deferred for 10 years, is, by Question VI, equal to 5·093: consequently the difference between these two values, or 7·204, will be the value required.

Had the estate depended on the life of his wife aged

40; then 7·475 (or the difference between 14·401 and 6·926) would be the value of the temporary annuity in this case required.

Or had the estate depended on their *joint* lives; then 6·775 (or the difference between 10·286 and 3·511) would be the value of the temporary annuity in this case required.

QUESTION VIII.

§ 385. To find the value of an annuity on the *Longest* of *two* lives.

SOLUTION.

From the sum of the values of an annuity on the two single lives, subtract the value of an annuity on the two joint lives: the difference will be the value required. See § 56.

Example 1. What is the value of an annuity on the longest of two lives aged 20 and 40; interest at $4\frac{1}{2}$ per cent and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on the two single lives is, by Table V, equal to 16·624 and 14·254 respectively, the sum of which is 30·878; therefore, if from this we subtract 12·545, or the value of an annuity on the two joint lives as found by Table IX, the difference, or 18·333, will be the value required.

Had the ages of the given lives been 50 and 70, the sum of the values of an annuity on their single lives would, by Table V, be equal to 18·142 (that is, equal to 11·921 added to 6·221); and the value of an annuity on their joint lives would, by Table IX, be equal to 5·517: consequently 12·625, or the difference between these two values, would be the value of an annuity on the longest of their two lives.

Had both the lives been 20 years of age, the value of an annuity on their single lives would (according to the same rate of interest &c) have been equal to twice 16·624; that is, equal to 33·248: and the value of an annuity on their joint lives would, by Table VI, be equal to 14·004: therefore the difference between these two values, or 19·244, would be the number of years purchase in this case required.

Example 2. What is the value of an annuity on the longest of two lives, a man and his wife, the former aged 46 and the latter aged 40: interest at 4 per cent, and the probabilities of living as observed in *Sweden?*

The value of an annuity on the life of the man is, by Table XVI, equal to 12·297, and the value of an annuity on the life of the woman is 14·401; the sum of these is 26·698: from which we must subtract 10·286, the value of an annuity on their joint lives by Table XVIII; and the difference, or 16·412 will be the value of an annuity on the longest of their two lives.

Had the two lives been each of them 10 years older, or 56 and 50 years of age, then the sum of the values of an annuity on their single lives would, by Table XVI, be equal to 21·766 (that is, equal to 9·717 added to 12·049), and the value of an annuity on their joint lives would, by Table XVIII, be equal to 7·874; consequently 13·892, or the difference between these two values, would be the value of an annuity on the longest of their lives.

Had both the lives been 40 years of age, then 10·964 (or the value of an annuity on their joint lives) subtracted from 28·069 (or the sum of the values of an annuity on their single lives*) would give 17·105 for the answer in this case required.

QUESTION IX.

§ 386. To find the value of an annuity on the *Longest* of *three* lives.

SOLUTION.

From the sum of the values of an annuity on all the single lives, subtract the sum of the values of an annuity on each *pair* of joint lives, and to the difference add the value of an annuity on the three joint lives: this last sum will be the value required. See § 56.

Example. What is the value of an annuity on the longest of three lives aged 20, 30, and 40; interest at 4 per cent, and the probabilities of living as at *Northampton?*

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* The value of an annuity on the life of the man is 13·668, and the value of an annuity on the life of the woman is 14·401.

The value of an annuity on each single life is, by Table XXVII, equal to 16·033, 14·781, and 13·197 respectively, the sum of which is 44·011; the value of an annuity on each pair of joint lives, (viz 20 and 30, 20 and 40, 30 and 40) is, by Tables XXX and XXXII, equal to 11·873, 10·924, and 10·490 respectively, the sum of which is 33·287; the difference between these two values is 10·724, which being added to 8·986, (or the value of an annuity on the three joint lives, as found by Table XLIV) will give 19·710 for the number of years purchase required.

Had all the three lives been 20 years of age, the value of an annuity on their single lives would have been equal to thrice 16·033, or 48·099; the value of an annuity on each pair of joint lives would have been equal to 37·605, or to thrice 12·535 (that is, equal to thrice the value of an annuity on two joint lives both aged 20, as found by Table XXVIII); and the value of an annuity on the three joint lives would, by Table XLIII, be equal to 10·342: therefore 20·836 would be the number of years purchase in this case required.

QUESTION X.

§ 387. To find the value of an annuity granted upon *three* lives, but to continue only as long as any *two* of them are in being together.

SOLUTION.

From the sum of the values of an annuity on each *pair* of joint lives, subtract twice the value of an annuity on the three joint lives, the difference will be the value required. See § 64.

Example. An annuity is purchased upon three lives aged 20, 30, and 40; on this condition, that as soon as any two of the lives fail, the annuity shall cease: the value of the same is required, reckoning interest at 4 per cent, and the probabilities of living as at *Northampton?*

The value of an annuity on each pair of joint lives (viz 20 and 30, 20 and 40, 30 and 40) is, by Table XXX and XXXII, equal to 11·873, 10·924, and 10·490 respectively, the sum of which is 33·287; and the value of an annuity on the three joint lives is 8·986: therefore twice the latter quantity, or 17·972, subtracted from 33·287 will give 15·315 for the number of years purchase required.

Had the ages of all the three lives been 20 years, the value would, in this case, have come out equal to 16·921.

QUESTION XI.

§ 388. To find the value of an annuity, on the *longest* of any number of lives, *Deferred* for any given term.

SOLUTION.

Substitute the values of *deferred* annuities on each single and joint lives, instead of the annuities for the *whole continuance* of those lives, and proceed as in the solutions to the two preceding questions. See § 60.

Example 1. What is the value of an annuity granted on the longest of two lives aged 20 and 40, but which is not to be entered on or enjoyed till after the expiration of 30 years: reckoning interest at 4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of a deferred annuity for 30 years on a life aged 20 is, by Question VI, equal to 2·272; the value of a similar annuity on a life of 40 is equal to ·784; and the value of a similar annuity on the two joint lives is equal to ·496: therefore if from the sum of the two former, or 3·056, we subtract the latter, the difference, or 2·560, will be the value required.

Example 2. A man and his wife (the former aged 46, and the latter aged 40) purchase on the longest of their two lives the reversion of the lease of an estate, which they are not to enter upon till the end of 10 years: what is the present value of the same, interest being reckoned at 4 per cent, and the probabilities of living as observed in *Sweden?*

The value of an annuity on the life of a male aged 46, deferred 10 years, is, by Question VI, equal to 5·093; the value of a similar annuity on a female aged 40, is, by the same question, equal to 6·926; and the value of a similar annuity on their joint lives is equal to 3·511. Consequently this latter value deducted from the sum of the two former ones, will leave 8·508 for the answer required.

§ 389. These examples give the present values in a *single* payment; but, if we wish to determine the same value in *annual* payments commencing immediately, we must divide the single payment thus found, by unity added to the value of an annuity on the longest of the given lives for one year less than the given term.

Thus in the second example, the value of the deferred annuity in a *single* payment is 8·508; and, by the rule in the following question,* the value of an annuity on the longest of the two lives for 9 years is equal to 7·251: consequently 8·508 divided by 8·251 will give 1·031 for the value in *annual* payments.

SCHOLIUM.

§ 390. It should here be particularly observed that if the deferred annuity, mentioned in this question, depends upon the *joint* existence of all the lives, to the end of the given term, the solution will be materially different; and these two cases must not be confounded. In this latter case the value will be equal to the value of an annuity on the longest of the same number of lives (each older by the given term than the given lives) multiplied by the expectation that the *joint* lives shall receive £1 at the end of that term. See § 61.

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* A more convenient method however of determining *such* temporary annuities is expressed by the following rule. To the value of the deferred annuity on the longest of the given lives add the expectation that the *longest* of such lives shall receive £1 at the end of the given term; subtract the sum from the value of an annuity on the longest of the given lives: the difference will be the value of the annuity for *one year less than the given term.* See the note in page 350.

Example 1. What is the value of an annuity on the longest of two lives aged 20 and 40, but which is not to be entered upon till the end of 30 years, and then only in case *both* the lives are in existence: interest at 4½ per cent, and the probabilities of life as observed by *M. De Parcieux?*

The value of an annuity on the longest of two lives aged 50 and 70 is, by the rule in Question VIII, equal to 12·625; and the expectation, that two lives aged 20 and 40 will receive £1 at the end of 30 years, is, by Question II, equal to ·0899: the product of these two quantities will give 1·135 for the answer required.

Example 2. A man (aged 46) and his wife (aged 40) purchase an annuity on the longest of their two lives, which is to commence at the end of 10 years provided they are *both* alive: what is the present value of the same, interest at 4 per cent, and the probabilities of life as observed in *Sweden?*

The value of an annuity on the longest of two lives (a man aged 56, and a woman aged 50) is, by the rule in Question VIII, equal to 13·892; and the expectation that two such lives aged 46 and 40 will receive £1 at the end of 10 years is, by Question II, equal to ·4459: the product of these two quantities will give 6·194 for the value required.

§ 391. The value of these annuities in *annual* payments commencing immediately will be equal to the value in a *single* payment, divided by unity added to the value of an annuity on the *joint* lives for one year less than the given term.

Thus, in the second example, the value of the deferred annuity in a *single* payment is equal to 6·194: and by the rule in the note to page 380, the value of an annuity on the two joint lives deferred for 9 years is 6·329: consequently 6·194 divided by 7·329 will give ·845 for the value in annual payments.

QUESTION XII.

§ 392. To find the value of a *Temporary* annuity on the *longest* of any number of lives.

SOLUTION.

From the absolute value of an annuity on the longest of the given lives, subtract the value of the same annuity *deferred* during the given term: the difference will be the value required.* See § 62.

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* Or (which is the same thing) substitute the values of *temporary* annuities on each single and joint lives, instead of the values of annuities for the *whole continuance* of those lives; and proceed as in the solutions to Questions VIII and IX.

Example 1. What is the value of a temporary annuity for 30 years on the longest of two lives aged 20 and 40; reckoning interest at $4\frac{1}{2}$ per cent, and the probabilities of life as given by *M. De Parcieux?*

The value of an annuity on the longest of those lives is, by Question VIII, equal to 18·333; and the value of an annuity on the longest of those lives *deferred* for 30 years is, by Question XI, equal to 2·560; consequently the difference between these two values, or 15·773, will be the answer required.

Example 2. A man aged 46 purchases an annuity for 10 years, terminable, however, at any time prior thereto, on the extinction of his own life and the life of his wife aged 40: what is the value of the same, interest at 4 per cent, and the probabilities of living as observed in *Sweden?*

The value of an annuity on the longest of their lives is, by Question VIII, equal to 16·412; and the value of an annuity on the longest of their lives *deferred* for 10 years is, by Question XI, equal to 8·508; the difference therefore between these two values, or 7·904, is the answer required.

QUESTION XIII.

§ 393. To find the value of the *Reversion* of an annuity on a *single* life after any other *single* life.*

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* This Question, and also Question XVIII, are of considerable importance in enabling us to determine the best means of providing Annuities for the *benefit of Widows;* as will be more fully explained in the following Chapter.

SOLUTION.

From the value of an annuity on the life in reversion, subtract the value of an annuity on the two joint lives; the difference will be the value required. See § 76.

Example 1. A person aged 20 wishes to purchase an annuity for what may happen to remain of his life beyond another life aged 40: what ought he to give for the same, allowing interest at 4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on the life in reversion (that is, on the life aged 20) is, by Table V, equal to 16·624; and the value of an annuity on the two joint lives is, by Table IX, equal to 12·545; therefore, the difference of these two values, or 4·079, is the number of years purchase required.

Had the life in reversion been 40, and the life in possession 20 years of age, the value would have come out equal to 1·709.

Or, had both the lives been 20 years, the value would have come out equal to 2·620: and had they both been 40 years of age, the value would have come out equal to 2·544.

Example 2. What is the value of an annuity to be enjoyed by a woman aged 40, during her life, after the decease of her husband aged 46; interest at 4 per cent, and the probabilities of living as amongst males and females respectively in *Sweden?*

The value of an annuity on the life of a woman aged 40 is, by Table XVI, equal to 14·401, and the value of an annuity on their joint lives is, by Table XVIII, equal to 10·286; therefore 4·115 is the number of years purchase required.

Had their lives been both 40 years of age, the value would have come out equal to 3·437.*

SCHOLIUM.

§ 394. If, instead of a *single* payment, we wish to determine the value of these reversionary annuities in *annual* payments to be made during the existence of the two joint lives; we must divide the value, found in either case, by unity added to the value of an annuity on the joint lives: and the quotient will give the annual payments required.

Thus, in the first example, 4·079 being divided by 13·545 will give ·301 for the *annual* payments which ought to be made during the joint lives, as an equivalent for the sum in a single payment. In like manner, in the second example, 4·115 being divided by 11·286 will give ·365 for the *annual* payments which ought to be made by a man aged 46 during the joint lives of himself and his wife, aged 40, in order to secure to his widow, on his death, an annuity of £1 per annum during her life.*

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* It is worthy of remark that the value of a reversionary annuity on one life after another is, when the difference of age is not very considerable, *less* in the younger ages and *greatest* in the middle ages of life: a circumstance which may be attributed to the higher chances of living in the younger ages, whereby the probability of *survivorship* is deferred so long as to affect in a material degree the value of the *reversionary* annuity.

QUESTION XIV.

§ 395. To find the value of the *Reversion* of an annuity on a *single* life A, after the *longest* of two other lives, P and Q.

SOLUTION.

From the sum of the values of an annuity on the single life A in reversion, and on the three joint lives, subtract the sum of the values of an annuity on the two joint lives AP and AQ: the difference will be the value required. See § 76.

Example. What is the value of an annuity on the life of a person aged 20 to be enjoyed by him after the decease of both his brother and sister, aged 30 and 40 respectively: interest at 4 per cent, and the probabilities of living as at *Northampton?*

The value of an annuity on the single life in reversion is, by Table XXVII, equal to 16·033, and the value of an annuity on the three joint lives is, by Table XLIV, equal to 8·986, the sum of which is 25·019; the value of an annuity on the two joint lives 20 and 30 is, by Table XXX, equal to 11·873, and the value of an annuity on the two joint lives 20 and 40 is, by Table XXXII, equal to 10·924, the sum of which is 22·797; therefore 22·797 subtracted from 25·019 will leave 2·222 for the value required.

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* Dr. Price has given a table of the value of reversionary annuities for the life of a wife after the death of her husband; both in *single* and *annual* payments during their joint lives: deduced from the *Sweden* observations and at 4 per cent interest, according to the several ages therein mentioned. See his *Obs. on Rev. Pay.* vol. ii. page 431. The utility and convenience of the present rule, in enabling us to determine the propriety and efficacy of those schemes which are instituted for the *benefit of Widows,* will be shown in the following Chapter.

Had the two lives in possession been both 40, then 16·033 added to 8·601 (or the value of an annuity on three joint lives aged 20, 40 and 40, as found by Question V) will make 24·634; from which we must subtract twice 10·924: the difference, or 2·786, will be the value in this case required.

QUESTION XV.

§ 396. To find the value of the *Reversion* of an annuity on the *longest* of two lives A and B, after any *single* life P.

SOLUTION.

From the sum of the values of an annuity on each single life A and B in reversion and on the three joint lives, subtract the sum of the values of an annuity on each pair of joint lives AB, AP, BP: the difference will be the value required. See § 76.

Example. What is the value of an annuity on the longest of two lives aged 20 and 30, to be enjoyed after the extinction of a single life aged 40; interest at 4 per cent, and the probabilities of living as at *Northampton?*

By proceeding as in the last question it will be found that the sum of the values of an annuity on the two single lives 20 and 30 is 30·814, that the value of an annuity on the three joint lives is 8·986, and that the sum of the values of an annuity on each pair of joint lives is 33·287; consequently, 6·513 is the value required.

Had the two lives in reversion been both 20 years of age, then the sum of the values of an annuity on their single lives would be 32·066; the value of an annuity on the three joint lives would, by Question V, be 9·406; and the sum of the values of an annuity on each pair of joint lives would be 34·383: consequently 7·089 would be the value required.

On the Renewal of Leases for Lives.

§ 397. The three preceding questions will be found of great practical use in the *Renewal of Leases*[*] held on two or three lives; as they serve to show the value of the *Fine* that ought to be paid for *putting in* a new life in lieu of one that has *dropt*, or become extinct. For, the value[†] of such fine is equal to the difference between the value of an annuity on the longest of all the lives (including the life or lives to be added) and the value of an annuity on the longest of the lives in possession: which rule will be found to agree with the solutions above given, according to the several cases there mentioned.*

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[*] See more on the subject of the Renewal of Leases for lives *and afterwards for a term certain,* in the observations at the end of Question XXIV.

[†] I call the *value* of a Fine, the *Number of years purchase* that it is worth: agreeably to the principles laid down in deducing the value of annuities: see the remark in the note in page 362. This value, being multiplied by the net improved rent of the estate, will show the total sum of money that ought to be given for the renewal.

Example 1. The value of the fine which ought to be given for *putting in* a new life, aged 20, to a lease held by Two lives, after One has dropped, is (supposing the existing life to be aged 40) equal to 4·079, or rather more than 4 years purchase of the net improved rent of the estate;† as already found by Question XIII.

Consequently, if the net improved rent of the estate had been £100 per annum, we should have £407 : 18 for the gross *sum* that ought to be paid down for the renewal required.

Example 2. The value of the fine which ought to be given for *putting in* a new life, aged 20, to a lease held by Three lives, after One has dropped, is (supposing the existing lives to be aged 30 and 40) equal to 2·222, or nearly $2\frac{1}{4}$ years purchase of the net improved rent of the estate; as already found by Question XIV.

Example 3. The value of the fine which ought to be given for *putting in* Two new lives, both aged 20, to a lease held by Three lives, after Two have dropped, is (suppoing the existing life to be aged 40) equal to 7·089, or rather more than 7 years purchase of the net improved rent of the estate; as already found by Question XV.

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* This subject is more fully discussed in my *Tables for the Purchasing and Renewing of Leases*, 2d edit. 1807.

† That is, the net *surplus* rent, after deducting the reserved rent (if any) and all taxes and other annual charges.

The same principles will also lead us to the true values that ought to be given for *Exchanging* any one or more lives (in a lease) for a life or lives of any other age. For, such value will in all cases be equal to the present value of the tenant's interest in the lease *before* the exchange, subtracted from his interest in the lease *after* the new lives are added.

QUESTION XVI.

§ 398. To find the value of the *Reversion* of an annuity on a *single* life, after the extinction of two *joint* lives.

SOLUTION.

From the value of an annuity on the single life in reversion, subtract the value of an annuity on the three joint lives: the difference will be the value required. See § 76.

Example. What is the value of an annuity on the life of a person aged 20 to be enjoyed by him after the decease of either of his brothers, one aged 30 and the other 40; interest at 4 per cent, and the probabilities of living as at *Northampton?*

The value of an annuity on the life of a person aged 20 is, by Table XXVII, equal to 16·033; and the value of an annuity on the three joint lives is, by Table XLIV, equal to 8·986: therefore 7·047 is the value required.

Had the two lives in possession been both 40 years, then 7·432 would have been the value required.

QUESTION XVII.

§ 399. To find the value of the *Reversion* of an annuity on two *joint* lives after the extinction of a *single* life.

SOLUTION.

From the value of an annuity on the two joint lives in reversion, subtract the value of an annuity on the three joint lives: the difference will be the value required. See § 76.

Example. What is the value of the reversion of an annuity on two joint lives aged 20 and 30 after the extinction of a life aged 40; interest at 4 per cent, and the probabilities of life as at *Northampton?*

The value of an annuity on the two joint lives aged 20 and 30 is, by Table XXX, equal to 11·873, and the value of an annuity on the three joint lives is, by Table XLIV, equal to 8·986: therefore 2·887 is the value required.

Had both the lives in reversion been 20 years of age, then 3·129 would be the answer required.

QUESTION XVIII.

§ 400. To find the value of any *Deferred reversionary* life annuity.*

SOLUTION.

Substitute the values of *deferred* annuities on each single and joint lives, instead of the annuities for the *whole continuance* of those lives; and proceed as in the solutions to the last five questions, according to the case. See § 77.

Example 1. What is the present value of a reversionary annuity on the life of a person aged 20, to commence at the end of 30 years, provided another person, now 40, be then dead; or if this should not happen, then at the end of any year in which the former shall happen to survive the latter: interest at 4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on the life in reversion, deferred for 30 years, is, by Question VI, equal to 2·272; and the value of an annuity on the two joint lives deferred for 30 years is, by the same Question, equal to ·496. Therefore the difference between these two values, or 1·776, will, by Question XIII, be the answer required.

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* This Question is of considerable use in enabling us to determine the validity of certain Schemes which have been proposed for providing Annuities for the *benefit of Widows;* as will be more fully explained in the following Chapter. See also the note in page 395.

Example 2. A woman aged 40 will at the end of 10 years enter upon an annuity for her life, provided her husband, now aged 46, be then dead; or if this should not be the case, then at the end of any year in which he may die: what is the present value of the reversion, interest at 4 per cent, and the probabilities of living as observed in *Sweden?*

The value of an annuity on the life of a female aged 40, deferred for 10 years, is, by Question VI, equal to 6·926; and the value of an annuity on their joint lives is, by the same Question, equal to 3·511. Therefore the difference between these two values, or 3·415, will, by Question XIII, be the answer required.

If we wish to determine the value of these deferred reversionary annuities in *annual* payments during the continuance of their joint lives, we have only to divide the *single* payment, above found, by unity added to the value of an annuity on the joint lives; as already explained in the Scholium to Question XIII.

SCHOLIUM.

§ 401. If the deferred annuity mentioned in this question depends on the *joint* continuance of all the lives to the end of the given term, the solution will be materially different, (as I have already observed respecting deferred annuities depending on the longest of any lives, in the Scholium in page 330): and care must be taken not to confound the two cases together. When the reversion depends on the *joint* continuance of all the lives to the end of the given term, its value will be equal to the value of the reversion on the same number of lives, each older by the given term than the given lives, multiplied by the expectation that the *joint* lives shall receive £1 at the end of that term. See § 78.

Example 1. What is the present value of a reversionary annuity on a life aged 20 for what may happen to remain of it beyond another life aged 40 after 30 years, provided *both* lives continue from the present time to the end of the term: interest at 4½ per cent, and the probabilities of living as given by *M. De Parcieux?*

The value of an annuity on a life aged 50 after another life aged 70 is, by the rule in Question XIII, equal to 6·404 (or the difference between 11·921 and 5·517); and the expectation that the joint lives, 20 and 40, shall receive £1 at the end of 30 years is, by Question II, equal to ·0899; the product of these two quantities therefore, or ·576, will be the value required.

Example 2. What is the present value of an annuity on the life of a woman aged 40, for what may happen to remain of it beyond the life of her husband, now aged 46, after 10 years, provided they *both* continue in being so long: interest at 4 per cent, and the probabilities of life as observed in *Sweden?*

The value of an annuity on the life of a woman aged 50 after the decease of her husband aged 56, is, by the rule in Question XIII, equal to 4·175; and the expectation that the joint lives 40 and 46 shall receive £1 at the end of 10 years is, by Question II, equal to ·4459: consequently, these two quantities multiplied together will produce 1·862 for the value in this case required.

If we wish to determine the value of these deferred reversionary annuities in *annual* payments during the continuance of their joint lives, we have only to divide the *single* payment, above found, by unity added to the value of an annuity on the joint lives; as already explained in the Scholium to Question XIII.

QUESTION XIX.

§ 402. To find the value of any *Temporary reversionary* life annuity.

SOLUTION.

Substitute the values of *temporary* annuities on each single and joint lives, instead of the annuities for the *whole continuance* of those lives; and proceed as in the last question. See § 77.

Example 1. A lease of an estate is held for 30 years, to the rent of which a person now aged 20 will be entitled on the decease of his brother aged 40: what is the value of his interest therein, taking the probabilities of life as observed by *M. De Parcieux*, and interest at $4\frac{1}{2}$ per cent?

The value of an annuity for 30 years on a life aged 20 is, by Question VII, equal to 14·352, and the value of a similar annuity on the two joint lives is, by the same Question, equal to 12·049; consequently, the difference of these two values, or 2·303, will, by Question XIII, be the answer required.

Example 2. In a lease of an estate (originally granted for twenty-one years) 10 years are unexpired; to the rent of which, a woman aged 40 will, on the decease of her husband aged 46, become entitled: what is the value of her interest in the same, taking the probabilities of life as observed in *Sweden*, and the rate of interest at 4 per cent?

The value of an annuity on the life of a woman aged 40 for 10 years is, by Question VII, equal to 7·475, and the value of a similar annuity on the two joint lives is, by the same Question, equal to 6·775: consequently ·700 is the value required.

If we wish to determine the value of these temporary reversionary annuities in *annual* payments, we have only to divide the single payments, above found, by unity added to the value of a temporary annuity on the joint lives *for one year less than the given term*: and the quotient will be the annual payment required. See § 368.

QUESTION XX.

§ 403. Two persons A and B purchase an annuity on the longest of their lives, which is to be equally divided between them whilst they are both living, but on the decease of either of them it is to belong wholly to the *Survivor:* to find their respective shares, or the proportion which each person ought to contribute towards the purchase.

SOLUTION.

From the value of an annuity on the life A or B subtract half the value of an annuity on the two joint lives: the remainder will be the share of A or B required. See § 85.

Example 1. Suppose the age of A to be 20, and that of B 40; the rate of interest 4½ per cent, and the probabilities of life as observed by *M. De Parcieux?*

The value of an annuity on a life aged 20 is, by Table V, equal to 16·624; and the value of an annuity on a life aged 40 is equal to 14·254: also the value of an annuity on two joint lives aged 20 and 40 is, by Table IX, equal to 12·545, the half of which is 6·272. Consequently this latter value subtracted from 16·624 will give 10·352 for the share which A ought to contribute; and subtracted from 14·254 it will give 7·982 for the share which B ought to contribute.

Example 2. Suppose two persons, a man aged 46 and a woman aged 40, to hold the lease of an estate on the longest of their lives, the rent of which is divided in the manner above stated; what sum ought to be given to each of them for surrendering their right in the same: interest being reckoned at 4 per cent, and the probabilities of life as observed in *Sweden?*

The value of the man's interest is equal to 7·154 (or to the difference between 12·297 and 5·143); and the value of the woman's interest is equal to 9·258 (or to the difference between 14·401 and 5·143). Therefore if the net rent of the estate were £50 per annum, the sum which ought to be given to the man will be 357·700, or £357 : 14 : and the sum which ought to be given to the woman will be 462·900, or £462 : 18.

SCHOLIUM.

§ 404. If the annuity is for a term of years *less* than that to which it is probable the given lives may extend, we must substitute the values of annuities for the given *term* instead of the values of annuities for the *whole continuance* of the lives: and proceed with these substituted values according to the directions given in the rule.

Thus, if in the first example, the annuity had been for 30 years only, we must find the value of a *temporary* annuity for 30 years on a single life aged 20, a single life aged 40, and two joint lives aged 20 and 40: which values are, by Question VII, equal to 14·352, 13·470, and 12·049 respectively. Consequently, the half of the latter subtracted from 14·352 will leave 8·328 for the share of A; and subtracted from 13·470, it will leave 7·446 for the share of B.

QUESTION XXI.

§ 405. Two persons are in possession of an annuity on the longest of their lives; which, on the decease of either of them, will belong to D and his heirs during the life of the *Survivor:* to find the value of his interest therein.

SOLUTION.

From the sum of the values of an annuity on each single life in possession, subtract twice the value of an annuity on their joint lives: the difference will be the value required. See § 137.

Example. Suppose the ages of the two lives in possession to be 20 and 40; interest 4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on each single life is, by Table V, equal to 16·624 and 14·254; and the value of an annuity on the two joint lives is, by Table IX, equal to 12·545. Consequently, 25·090 subtracted from 30·878 will leave 5·788 for the interest of D and his heirs in this annuity.

Had the two lives in possession been a man aged 46 and a woman aged 40, the value of the interest of

D and his heirs in the annuity would (on the supposition that interest was at 4 per cent, and the probabilities of living as observed in *Sweden*) be equal to 6·126.

SCHOLIUM.

§ 406. If the annuity is for a term of years *less* than that to which it is possible that either of the given lives may extend, we must substitute the values of annuities for the given *term* instead of the values of annuities for the *whole continuance* of the lives: and proceed with these substituted values according to the directions given in the rule.

Thus, if in the example just given, the annuity on the two lives aged 20 and 40 had been for 30 years only, then 24·098 (or twice the value of a temporary annuity for 30 years on the two joint lives, as found by Question VII) subtracted from 27·822 (or the sum of the values of a temporary annuity for 30 years on each of the single lives, as found by the same Question) will leave 3·724 for the value in this case required.

QUESTION XXII.

§ 407. To find the value of an annuity *certain* for a given term; and afterwards, for the *remainder* of any given life or lives.

SOLUTION.

To the present value of an annuity certain for the given term, add the value of an annuity on the given life, or lives, *deferred* for that term: the sum of these two will be the value required. See § 51.

Example 1. What is the value of an annuity *certain* for 30 years, and then to continue during the life of a person now aged 20: reckoning interest at 4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity certain for 30 years is, by Table LIX, equal to 16·289, and the value of an annuity, on a life aged 20, deferred for 30 years is, by Question VI, equal to 2·272: consequently 18·561 will be the value required.

Example 2. What is the value of an annuity *certain* for 10 years, and then to continue during the joint lives of a man aged 46 and his wife aged 40: reckoning interest at 4 per cent, and the probabilities of living as observed in *Sweden?*

The value of an annuity certain for 10 years is, by Table LIX, equal to 8·111, and the value of an annuity on the two joint lives, deferred for 10 years, is, by Question VI, equal to 3·511: consequently 11·622 will be the value required.

QUESTION XXIII.

§ 408. Supposing a person to enjoy an annuity for his life; and, at his decease, to have the nomination of a successor: to find the present value of the annuity on the *Succeeding life.*

SOLUTION.

Multiply the value of an annuity on the life in possession by the rate of interest, and subtract the product from unity; multiply the remainder by the assumed* value of an annuity on the succeeding life: the product will be the present value required. See § 205.

Example. Suppose the age of the life in possession to be 65, and that at his decease he has the liberty of nominating another life to succeed him; which life we will suppose to be one of the best that can *then* be found, or one which may *then* be about 10 years old: what is the *present* value of such succeeding life, interest at 5 per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on a life aged 65 is, by Table V, equal to 7·535, which being multiplied by ·05, and subtracted from unity, will leave ·62325; and this quantity, multiplied by 16·213 (or the value of an annuity on the life to be hereafter nominated) will produce 10·105 for the *present* value of the same, as was required.

Had it been required to calculate the value of the

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[*] The life or lives nominated to succeed to the annuity, after the extinction of the life or lives in possession, are such as are *then* to be fixed on, at pleasure: therefore the *present* value of an annuity on those lives will vary according to the ages at which they are supposed to be put in. See page 163.

succeeding life according to the *Northampton* tables of observations, at the same rate of interest; we ought to multiply 7·276 by ·05; which, subtracted from unity, would leave ·63620; and this latter quantity multiplied by 15·139* will give 9·631 for the value in this case required.

SCHOLIUM.

§ 409. The solution given to the present question will apply equally to the case of annuities on *joint* lives, or on the *longest* of any lives; with power to nominate, at the extinction of such lives, an equal number of similar lives to succeed thereto.

Example. Suppose an annuity is held on two *joint* lives aged 50 and 60; and that, on the extinction of either of them, two other joint lives (the best that can then be found, and which we will suppose to be 10 years old) are nominated to succeed them: what is the present value of the annuity on the succeeding joint lives, interest at 5 per cent, and the probabilities of living as observed at *Northampton?*

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* It appears that, by the *Northampton* tables, a life of the age of 8 years is one of the best that can be put in; since the value of an annuity on such life is equal to 15·226: but in questions of this kind we may safely omit the *decimal,* and assume the life to be such that an annuity on it at the time of nomination would be worth 15 years purchase. For, it seldom happens that the life, which we should *choose* to nominate, is exactly of the age which is assumed.

The value of an annuity on the two *joint* lives is, by Table XXX, equal to 6·568; which, being multiplied by ·05 and subtracted from unity, will leave ·67160; and this multiplied by 12·665 (or the value of an annuity on two joint lives both 10 years old) will produce 8·506 for the value required.

Had the annuity been held on the *longest* of the two lives, aged 50 and 60; with power, on the extinction of *both* those lives, to nominate two other lives (whose ages we will suppose to be each 10 years) who are to enjoy the annuity as long as either of them is in existence; the present value of the annuity on those succeeding lives might be calculated in a similar manner. For, the value of an annuity on the longest of two lives aged 50 and 60 is, by the solution in Question VIII, equal to 12·093; which being multiplied by ·05 and subtracted from unity, leaves ·39535; and this quantity multiplied by 17·613 (or the value of an annuity on the longest of two lives aged 10 years) will give 6·963 for the value required.

QUESTION XXIV.

§ 410. To find the present value of an annuity *certain* for a given term *after* the extinction of any given life or lives.

SOLUTION.

Multiply the value of an annuity on the given life or lives by the rate of interest, and subtract the product from unity; multiply the remainder by the present value of an annuity certain for the given term: the product will be the value required. See § 208.

Example 1. Suppose D and his heirs to be entitled to an annuity certain for 21 years, to commence at the death of a person aged 70; what is the present value of D's interest in that annuity, taking the probabilities of living as observed at *Northampton*, and the rate of interest at 5 per cent?

The value of an annuity on the life of a person aged 70 is, by Table XXVII, equal to 6·023, which being multiplied by ·05 and subtracted from unity, will leave ·69885; and this multiplied by 12·821 (or the value of an annuity certain for 21 years) will produce 8·960 for the present value of the same annuity to be entered on* at the extinction of the given life.

If this value be added to 6·023 (or the value of the annuity on the life in possession) the sum of them, or 14·983, will be equal to the value of an annuity on the given life, commencing immediately, and to continue, after the extinction of such life, for the term of 21 years longer.

Example 2. A lease of an estate is held upon two lives aged 60 and 70; and, after the decease of both of them, then for 21 years longer: what is the value of such lease, reckoning interest at 5 per cent, and the probabilities of living as at *Northampton?*

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* This solution supposes that the first payment of the annuity is made at the end of the year in which the given life becomes extinct.

The value of an annuity on the longest of two lives aged 60 and 70 is, by the rule in Question VIII, equal to 9·735; which, being multiplied by ·05 and subtracted from unity, will leave ·51325; this being multiplied by 12·821 (or the value of an annuity certain for 21 years) will give 6·580 for the present value of the same annuity to be enjoyed 21 years after the extinction of the longest of the two lives. And this value being added to 9·735 (or the value of an annuity on the longest of the two lives) will give 16·315 for the value of the lease required.

Example 3. A lease of an estate is held upon three lives aged 50, 60, and 70; and, after their decease, then for 21 years longer: what is the value of the same, reckoning interest at 5 per cent, and the probabilities of living as at *Northampton?*

The value of an annuity on the longest of three lives aged 50, 60, and 70 is, by the rule in Question IX, equal to 12·443;* which, being multiplied by ·05 and then subtracted from unity, will leave ·37785; this being multiplied by 12·821 will give 4·844 for the present value of the annuity for 21 years after the longest of the three lives. And this value, being added to 12·443, will give 17·287 for the value of the lease required.

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* The value of an annuity on the three *joint* lives is, by the rule in Question V (and the correction alluded to in page 376) equal to 4·061.

Had the three lives been 10, 60, and 70 years of age; then the value of an annuity on the longest of their lives would be equal to 15·878;* and the value of the annuity for 21 years after those lives would be equal to 2·642: consequently the value of the lease would in this case be equal to 18·520.

Or, had the three lives been 10, 10 and 70 years of age; then the value of an annuity on the longest of their lives would be equal to 17·610;† and the value of the annuity for 21 years after those lives would be equal to 1·532: consequently the value of the lease would in this case be equal to 19·142.

On the Renewal of Leases for Lives and afterwards for a Term certain.

§ 411. The three examples given in the preceding question will serve to show the method of determining the value of the *Fine* which ought to be given for *Renewing* any lives dropt in a lease originally held on three lives *and for a term certain after the extinction of those lives.* For, the value of such fine will in all cases be equal to the present value of the tenant's interest in the lease *before* the renewal, subtracted

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[*] The value of an annuity on these three *joint* lives is, by the rule in Question V (and the correction alluded to in page 376) equal to 4·454.

[†] The value of an annuity on these three *joint* lives is, by the rule in Question V (and the correction alluded to in page 376) equal to 5·374.

from his interest in the lease *after* the new lives are added.*

Thus, suppose that, in a lease originally held on three lives and 21 years, *one* of the lives has dropt; and that the ages of the two remaining lives are 60 and 70: the value of the fine which ought to be paid for putting in another life aged 10 years is equal to the difference between 16·315 (or the value found by the second example) and 18·520 (or the value found by the second case in the third example). That is, the value of the fine will be equal to 2·205, or about $2\frac{1}{5}$ years purchase of the net improved rent of the estate.

Again, let us suppose that *two* of the lives have dropt; and that the age of the remaining life is 70: the value of the fine which ought to be paid for putting in two other lives both 10 years of age, is equal to the difference between 14·983 (or the value found by the first example) and 19·142 (or the value found by the third case in the third example): consequently the value of the fine will be equal to 4·159, or about $4\frac{1}{6}$ years purchase of the net improved rent of the estate.

These examples will also serve to show the sum that ought to be given for *Exchanging* any of the lives on which the lease may happen to be held: for, the same method of solution will apply to such cases.

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* See what has been already said on the subject of the Renewal of Leases for Lives in general, in page 307.

Thus, suppose that, in a lease held on three lives and 21 years, the ages of the lives at present in the lease were 50, 60, and 70; and that the tenant is desirous of exchanging the life of 50 for another life aged 10 years old: the value of the fine which ought in such case to be paid will be equal to the difference between 17·287 (or the value of his *present* interest, as found by the first case in the third example) and 18·520 (or the value of his interest *after* the exchange, as found by the second case in the same example). That is, the value of the fine will be 1·233, or near $1\frac{1}{4}$ year's purchase of the net improved rent of the estate.

§ 412. Many of the estates belonging to the Corporation of *Liverpool* are held on the tenure alluded to in these examples; and till lately they were in the constant habit of renewing their leases on the following terms: viz. One year's purchase for adding *one* life dropt, Three years purchase for adding *two* lives dropt, and Seven years purchase for adding *three* lives dropt when the 21 years remain unexpired. In all these cases no regard was paid to the *age* or state of health of the existing lives in the lease. This practice of demanding an uniform fine for renewing with *any* life, and without regard to the age or state of health of the lives remaining in the lease, betrayed a total want of knowledge on the subject; and was in most cases injurious to the interests of the Corporation.

But, the most singular circumstance attending this subject was their custom of *exchanging*, for the sum of only *one guinea* each, lives, not exceeding 50 years of age and in good health, for lives of *any other age*, and in estates of *any* yearly value!!! A practice which could hardly be supposed ever to have existed in so enlightened a place as *Liverpool*. The Corporation at length, suspecting that their mode of proceeding was incorrect in principle, referred the matter to a Committee, who directed it to be laid before me for my opinion: and agreeably to their request I calculated a set of Tables for their use, founded on the principles detailed in the preceding examples.

As it is probable that many other Corporate bodies are still pursuing the same incorrect and absurd practice of leasing their estates, I have been more particular in these examples, in order that they may the more readily determine the values that *ought* to be given in such cases.

QUESTION XXV.

§ 413. To find the present value of what may happen to *remain* (after any given life, or lives) of an annuity *certain* for a given term: provided such term be *less* than that to which it is possible the given lives may extend.*

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* The term to which it is possible that any given life or lives may extend is—for a *single* life, equal to the difference between the age of such life and the age of the oldest life in the table of observations:—for *joint* lives, equal to the difference between the *oldest* of such lives and the age of the oldest life in the table of

SOLUTION.

From the value of an annuity certain for the given term, subtract the value of an annuity on the given life, or lives, *for the given term:* the difference will be the value required. See § 192.

Example. A lease of an estate is held for 30 years, to the rent of which a person aged 20 is entitled provided he lives so long; but if not, then the remainder of the lease will descend to his heirs: what is the value of their interest in the same, taking the probabilities of living as observed by *M. De Parcieux*, and reckoning interest at the rate of 4½ per cent?

The value of an annuity certain for 30 years is, by Table LIX, equal to 16·289; and the value of a temporary annuity for 30 years on a life aged 20 is, by Question VII, equal to 14·352: therefore this latter quantity subtracted from the former will leave 1·937 for the value of the reversion required.

Had the life been 40 years of age, the value would have been equal to the difference between 16·289 and 13·470: that is, 2·819 would be the value of the reversion in this case required.

Had these two *joint* lives (aged 20 and 40) been entitled to the rent of the estate provided they lived so

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observations:—and for the *longest* of any number of lives, equal to the difference between the *youngest* of such lives and the age of the oldest life in the table of observations.

long, then 12·049 (or the value of a temporary annuity for 30 years on those two *joint* lives, as found by Question VII) subtracted from 16·289, would give 4·240 for the value of the reversion in this case required.

Or, had the *longest* of these two lives been entitled to the rent of the estate, then 15·773 (or the value of a temporary annuity for 30 years on the *longest* of those lives, as found by Question XII) subtracted from 16·289, would give ·516 for the value of the reversion in this case required.

QUESTION XXVI.

§ 414. To find the value of the *Assurance* of an *estate* (or annuity certain for any given term*) to be entered upon at the extinction† of any given lives.

SOLUTION.

Subtract the value of an annuity on the given lives‡ from the value of the perpetuity, or the terminable annuity; and the difference will be the value required. See § 189.

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* Provided such term be *not less* than that to which it is probable the given lives may extend. For, in *such* case, the solution is obtained by the preceding Question.

† That is, the first payment of the annuity is to be made at the end of the year in which such lives become extinct: and this is always understood in questions of this kind.

‡ Whether a *single* life, or *joint* lives, or the *longest* of any number of lives: for the solution will apply to each case.

Example 1. What is the value of the reversion of an estate in fee after the death of a person now aged 20; interest being reckoned at 4½ per cent, and the probabilities of life as observed by *M. De Parcieux?*

The value of the perpetuity is, by Table LIX, equal to 22·222, and the value of an annuity on the life of a person aged 20 is, by Table V, equal to 16·624; consequently the difference between these two values, or 5·598, will be the answer required. Therefore if the estate produced a rent of £4 : 10 per annum, its present value in a *single* payment would be 25·191, or £25 : 3 : 10.

This is the true present value of the assurance in a *single* payment: but, in order to obtain the value of the same in *annual* payments, commencing immediately, we must divide the sum, thus found, by unity added to the value of an annuity on the given life (agreeably to the principles laid down in § 369): and the quotient will be the answer required.

Thus, in the present case, if we divide 25·191 by 17·624, the quotient will be 1·429, or £1 : 8 : 7; and this is the sum that ought to be paid annually during the life of the person assured, in order to secure the perpetuity of £4 : 10 per annum on his death: the first of those annual payments being made *immediately*, and the remaining ones at the *beginning* of every subsequent year.

Had the question referred to an *annuity* for 80 years,* instead of a perpetuity, then 16·624 subtracted from 21·565 (or the present value of an annuity certain for that term by Table LIX) would leave 4·941 for the answer required. Therefore, if the annuity, as in the preceding case, were £4 : 10 per annum, its present value in a *single* payment would be 22·234 : and this sum, divided by 17·624, would give 1·262 for the value of the same in *annual* payments.

Example 2. What is the value of a freehold estate to be entered upon at the death of *either* of two lives, a man aged 46 and a woman aged 40; reckoning interest at 4 per cent, and the probabilities of life as observed in *Sweden?*

The value of the perpetuity is, by Table LIX, equal to 25, and the value of an annuity on the joint lives of these two persons is, by Table XVI, equal to 10·286: this latter value subtracted from the former will leave 14·714 for the answer required. Therefore if the estate produced £4 per annum, its present value would be 58·856, or £58 : 17 : 1, in a *single* payment: and this sum, divided by 11·286, would give 5·215, or £5 : 4 : 4, for the value of the same in *annual* payments.

If the estate were not to be entered upon till the extinction of *both* the lives, then 16·412 (or the value of an annuity on the *longest* of the two lives, as found by Question VIII) subtracted from 25, will leave 8·588 for the number of years purchase required; and which being multiplied by 4, as in the preceding case, will give 34·352 for the value of the same estate in a *single* payment: and this sum divided by 17·412 will give 1·973 for the value of the same in *annual* payments.

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* It must be particularly observed that, when we have to determine by this rule the value of the reversion of any terminable annuity after any given lives, the number of years during which such annuity is to continue must not be *less* than that to which it is probable the given lives may extend: see the note in page 419.

Had it been a leasehold estate of £4 per annum for 60 years, instead of a freehold, the value would in the former case have come out equal to 49·348 in a *single* payment; or 4·373 in *annual* payments. And in the latter case, to 24·844 in a *single* payment; or 1·427 in *annual* payments.

QUESTION XXVII.

§ 415. To find the value of an *Assurance* of a given *sum*, which is to be received on the extinction* of any given lives.

SOLUTION.

Multiply the value of an annuity on the given lives† by the rate of interest, and subtract the product from unity; divide the remainder by the amount of £1 in one year: and the quotient, multiplied by the given sum, will be the value required. See § 180.

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* That is, at the end of the year in which such lives become extinct: and this is always understood in questions of this kind. The usual practice of the Assurance Offices, however, is to pay the sum at the end of six months from the time of the decease.

† Whether a *single* life, or *joint* lives, or the *longest* of any number of lives: for, the solution (as in the preceding question) will apply to each case.

Example 1. What is the present value of an assurance of £100 on the life of a person aged 20: interest at 4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on such life is, by Table V, equal to 16·624, which being multiplied by ·045 (or the rate of interest) will produce ·74808; the difference between this value and unity is ·25192, which being divided by 1·045 (or the amount of £1 in one year) will give ·24107 for the present value of £1 to be received on the extinction of the given life:* and this value, being multiplied by 100, will give 24·107, or £24 : 2 : 2, for the answer required, in a *single* payment.

But, in order to obtain the value of the same in

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* From the present value of *one pound* to be received on the extinction of any given life or lives, we may readily determine the sum, which ought to be paid, on the extinction of such lives, for *any given sum now advanced:* viz. by dividing this latter sum by the present value of £1 as above found. Thus, if a person, aged 20, borrows £4000, and gives security to pay the value of the same at his death, the sum which ought *then* to be paid (supposing the interest &c the same as mentioned in the text) is found by dividing 4000 by ·24107; which gives 16592·691, or £16592 : 13 : 10, for the answer required. This method is universal; and will apply to all the subsequent questions in the present Chapter.

annual payments, commencing immediately, we must divide the sum, thus found, by unity added to the value of an annuity on the given life; agreeably to the principles laid down in § 369: whence 24·107 divided by 17·624 will give 1·368, or £1 : 7 : 4, for the sum which ought to be paid annually during the life of the person assured, in order to secure the sum of £100 on his death. The first of those annual payments being made *immediately*, and the remaining ones at the *beginning* of every subsequent year.*

Had the life been 40 years of age, its value in a *single* payment would have been equal to 34·313: which being divided by 15·254 (or unity added to the value of an annuity on the given life) will give 2·249, or £2 : 5 : 0, for the value of the same in *annual* payments.

Example 2. What is the present value of £100 to be received on the death of a man aged 46: interest being reckoned at 4 per cent, and the probabilities of life as observed in *Sweden?*

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* The rates of Assurances for Lives, at *all* the different Offices established in London, are calculated from the *Northampton* Table of Observations, and at 3 per cent interest. These rates, both for *Single* and *Joint* lives, are given in Tables LI and LII at the end of the present work: and by comparing them with the true and proper values, as given in Tables LIV to LVII, the public may form a tolerably accurate idea of the immense profit which is made by the several Assurance Companies above alluded to. I shall however enlarge more upon this subject in Chapter XIV. See also the Scholium to Question XXIX, and the Scholium to Question XXX.

The value of an annuity on the life of a man aged 46 is, by Table XVI, equal to 12·297; which, being multiplied by ·04, and the product subtracted from unity, leaves ·50812: this quantity, divided by 1·04, gives ·48858; which, being multiplied by 100, produces 48·858, or £48 : 17 : 2, for the answer required, in a *single* payment. And if this latter sum be divided by 13·297, it will give 3·674, or £3 : 13 : 6, for the value of the same in *annual* payments. Had the age of the man been 56, the value of the assurance in a *single* payment would have been equal to 58·781; which divided by 10·717 would give 5·485 for the value of the same in *annual* payments.

But if the sum had depended on the death of a woman aged 40, its value in a *single* payment would have been equal to 40·765, or £40 : 15 : 4. And this sum, divided by 15·401, would give 2·647, or £2 : 12 : 11, for the value of the same in *annual* payments. And had the age of the woman been 50, the value of the assurance in a *single* payment would have been equal to 49·812; which divided by 13·049 would give 3·817 for the value of the same in *annual* payments.

Example 3. What sum ought to be given for the assurance of £100 on two *joint* lives aged 20 and 40: interest at 4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on the two joint lives is, by Table IX, equal to 12·545, which being multiplied by ·045 (or the rate of interest) will produce ·56452;

this quantity, subtracted from unity, leaves ·43547, which being divided by 1·045 (or the amount of £1 in a year) will give ·41672; and this multiplied by 100 will produce 41·672, or £41 : 13 : 5, for the answer required, in a *single* payment. If this latter quantity be divided by 13·545, it will give 3·077, or £3 : 1 : 6, for the value of the same in *annual* payments.

Had the two lives been 50 and 70, the value in a *single* payment would have been equal to 71·936; which being divided by 6·517 would give 11·035 for the value of the same in *annual* payments.

Had the assurance been made on the joint lives of a man aged 46, and his wife aged 40, the value of the same (reckoning interest at 4 per cent, and the probabilities of life as observed in *Sweden*) would have been equal to 56·592 in a *single* payment: and which, being divided by 11·286, would give 5·014 for the value in *annual* payments.

Or, had these two lives been respectively 56 and 50, the value would in such case have come out equal to 65·869 in a *single* payment: and which being divided by 8·874 would give 7·423 for the value in *annual* payments.

Example 4. What is the value of an assurance of £100 on the *longest* of two lives aged 20 and 40; interest $4\frac{1}{2}$ per cent, and the probabilities of life as observed by *M. De Parcieux?*

The value of an annuity on the longest of two lives aged 20 and 40 is, by Question VIII, equal to 18·333;

which, being multiplied by ·045 and subtracted from unity, leaves ·17502; this quantity, divided by 1·045, will give ·16748, and which being multiplied by 100 will produce 16·748, or £16 : 15 : 0, for the answer required, in a *single* payment. If this latter quantity be divided by 19·333 (or unity added to the value of an annuity on the *longest* of the two lives) it will give ·866 or 17*s* : 4*d* for the value of the same in *annual* payments, to be made at the beginning of each year during the continuance of either of the given lives.

Had the two lives been 50 and 70 years of age, the value in a *single* payment would have been equal to 41·328; which, being divided by 13·625, would give 3·033 for the value of the same in *annual* payments.

Or, had the assurance been made on the longest of the two lives of a man aged 46 and his wife aged 40, the value of the same (reckoning interest at 4 per cent, and the probabilities of life as observed in *Sweden*) would have been equal to 33·031 in a *single* payment: and which, being divided by 17·412, would give 1·897 for the value of the same in *annual* payments.

And had these two lives been respectively 56 and 50, the value would in such case have come out equal to 42·723 in a *single* payment: which being divided by 14·892 will give 2·869 for the value of the same in *annual* payments.

Example 5. What is the present value of a legacy* of £100 to be received on the extinction of any *one* of three lives aged 20, 30, and 40: reckoning interest at 4 per cent, and the probabilities of living as at *Northampton?*

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* I consider a legacy as not due till the *end* of the year in which the testator dies: for it is seldom paid *immediately*.

The value of an annuity on the three joint lives is, by Table XLIV, equal to 8·986, which, being multiplied by ·04, and the product subtracted from unity, leaves ·64056: this quantity divided by 1·04 will give ·61592, and which being multiplied by 100 will produce 61·592, or £61 : 11 : 10, for the present value of the legacy required.

In like manner we might determine the value of the legacy payable on the extinction of any *two* of the three lives above mentioned. For, the value of an annuity on any two out of those three lives is, by Question X, equal to 15·315: consequently, by proceeding as in the last case, we shall find that 37·250, or £37 : 5 : 0, will be the present value of the legacy in this case required.

So also we might find the value of the legacy payable on the extinction of all the *three* lives. For, the value of an annuity on the longest of the lives is, by Question IX, equal to 19·710: and by proceeding in a similar manner it will be found that 20·346, or £20 : 6 : 11, is the present value of the legacy in this case required.

SCHOLIUM.

§ 416. It may be here necessary to advert to the remark which I have made, in the Scholium in page 160, respecting the relative values of a reversionary *sum*, and a corresponding reversionary *estate:* and which may be verified by a comparison of the values in any two similar cases found by the rules in the two preceding Questions. Thus, by the first example in Question XXVII, it appears that the value of £100 payable on the decease of a person aged 20, interest at 4½ per cent, is equal to 22·234 pounds: and by the first example in Question XXVI, it appears that the value of a corresponding estate (or perpetuity) of £4 : 10 per annum is equal to 25·191 pounds. But the latter is to the former value, in the proportion of 1·045 to 1; and *vice versâ*, the former is to the latter value, as 1 is to 1·045.

QUESTION XXVIII.

§ 417. To find the value of a *Deferred assurance* of any given sum, on any given lives.

SOLUTION FIRST.

For Single and Joint Lives.

Find the value of the assurance of the given sum on the same number of single or joint lives as the given lives, but each older than such lives by the given term; multiply this value by the expectation that the given single or joint lives will receive £1 at the end of that term: the product multiplied by the given sum will be the value required. See the note in page 147.

Example 1. What is the present value of £100 to be received on the death of a man aged 46, provided that should happen *after* 10 years: interest being reckoned at 4 per cent, and the probabilities of life as observed in *Sweden?*

The value of an assurance of £100 on a man aged 56 is, by Question XXVII, equal to 58·781; and the expectation that a man aged 46 will receive £1 at the end of 10 years is, by Question II, equal to ·5241; consequently these two quantities being multiplied together will produce 30·807, or £30 : 16 : 2, for the answer required.

Had the assurance been on a woman aged 40, its present value would have been equal to 49·812 multiplied by ·5748; that is, equal to 28·632.

Example 2. What is the present value of £100 to be received on the extinction of *either* of two lives aged 20 and 40, provided that should happen *after* 30 years: interest at 4½ per cent, and the probabilities of life as observed by *M. De Parcieux?*

The value of the assurance on two joint lives aged 50 and 70 is, by Question XXVII, equal to 71·936; and the expectation of two joint lives, aged 20 and 40, receiving £1 at the end of 30 years is, by Question II, equal to ·0899: therefore the product of these two quantities will give 6·467, or £6 : 9 : 4, for the answer required.

Had the assurance been made on the joint lives of a man aged 46 and his wife aged 40, provided they became extinct *after* 10 years, its present value (taking interest at 4 per cent, and the probabilities of life as observed in Sweden) would have been equal to 65·869 multiplied by ·4459; that is, equal to 29·371.

SOLUTION SECOND.

For the Longest of any lives.

§ 418. Multiply the value of an annuity on the longest of the given lives, *deferred* for the given term, by the rate of interest; subtract the product from the expectation of the longest of the given lives receiving £1 at the end of the same period, and divide the difference by the amount of £1 in a year: the quotient multiplied by the given sum will be the answer required. See § 182.

Example 3. What is the value of £100 to be received on the extinction of the longest of two lives, aged 20 and 40, provided that shall happen *after* 30 years: interest at 4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of a deferred annuity for 30 years on the longest of the two lives is, by Question XI, equal to 2·560, which being multiplied by ·045 produces ·1152; and this subtracted from ·2266 (or the expectation that one or other of these two lives will receive £1 at the end of 30 years, as found by Question II) will leave ·1114, which being divided by 1·045 will give ·10660; and this last value, multiplied by 100, will produce 10·660, or £10 : 13 : 2, for the answer required.

Had this sum depended on the extinction of the longest of two lives, a man aged 46 and a woman aged 40, provided that event happened *after* 10 years, the value of the same (reckoning interest at 4 per cent, and the probabilities of life as observed in *Sweden*) may be found in a similar manner. For, the value of a deferred annuity for 10 years on the longest of these lives is, by Question XI, equal to 8·508, which being multiplied by ·04 produces ·3403; and this subtracted from ·6530 (or the expectation that one at least of the two lives will receive £1 at the end of 10 years, as found by Question II) will leave ·3127, which being divided by 1·04 will give ·30067; and this last value, multiplied by 100, will produce 30·067, or £30 : 1 : 4, for the answer required.

QUESTION XXIX.

§ 419. To find the value of a *Temporary assurance* of a given sum on any given lives.

SOLUTION.

From the value of the assurance of the given sum on the *whole continuance* of the given lives, subtract the value of a *deferred* assurance of the same sum for the given term: the difference will be the value required. See § 184.

Example 1. What is the present value of £100 to be received on the death of a man aged 46, provided that shall happen *within* 10 years: interest being reckoned at 4 per cent, and the probabilities of life as observed in *Sweden?*

The value of an assurance of £100 on the whole continuance of this life is, by Question XXVII, equal to 48·358; and the value of a similar assurance, deferred for 10 years is, by Question XXVIII, equal to 30·807, which being subtracted from the former value will leave 18·051, or £18 : 1 : 0, for the answer required.

This is the value in a *single* payment: but if we wish to find the corresponding value in *annual* payments, we must divide this sum by 7·680 (or unity added to the value of an annuity on the given life for one year less than the given term, as found by the rule given in the note, page 380), which will give 2·350, or £2 : 7 : 0, for the value of the same assurance in *annual* payments.

Had the assurance been made on a woman aged 40, its present value would have been equal to the difference between 40·765 and 28·632: that is, 12·133, or £12 : 2 : 8, would have been the answer required, in a *single* payment. And this value divided by 7·900 (or unity added to the value of an annuity on the life for 9 years, as found by the rule above mentioned) will give 1·536, or £1 : 10 : 9, for the value of the same in *annual* payments.

Example 2. What is the value of an assurance of £100 on the *joint* lives of two persons aged 20 and

40, for 30 years: interest at $4\frac{1}{2}$ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of the assurance on the whole continuance of the lives is, by Question XXVIII, equal to 41·672; and the value of a similar assurance deferred for 30 years is, by Question XXVIII, equal to 6·467: therefore this latter value, subtracted from the former, will leave 35 205, or £35 : 4 : 1, for the answer required, in a *single* payment. And this sum divided by 12·959 (or unity added to the value of an annuity on the two joint lives for twenty-nine years) will give 2·717, or £2 : 14 : 4, for the value of the same assurance in *annual* payments.

Had the assurance been made for 10 years on the joint lives of a man aged 46 and his wife aged 40, the value of the same (reckoning interest at 4 per cent, and the probabilities of living as observed in *Sweden*) would have been equal to the difference between 56·592 and 29·371: that is, 27·221 would have been the value in a *single* payment. And this sum divided by 7·329 (or unity added to the value of an annuity on the two joint lives for nine years) will give 3·714 for the value of the *annual* payments.

Example 3. What sum ought to be given for the assurance of £100 for 30 years on the *longest* of two lives, aged 20 and 40: interest at $4\frac{1}{2}$ per cent, and the probabilities of life as observed by *M. De Parcieux?*

The value of the assurance for the whole continuance of the given lives is, by Question XXVII, equal to

16·748; and the value of a similar assurance deferred for 30 years is, by Question XXVIII, equal to 10·660: this latter value therefore subtracted from the former will leave 6·088, or £6 : 1 : 9, for the answer required, in a *single* payment. And this sum divided by 16·546 (or unity added to the value of an annuity on the longest of the two lives for twenty-nine years) will give ·368, or 7*s* : 4*d*, for the value of the same in *annual* payments.

Had it been required to determine the value of the assurance, in annual payments during the *joint* continuance of the given lives, then 6·086 divided by 12·959 (or unity added to the value of an annuity on the two joint lives for twenty-nine years) would give ·470 for the answer required.

In like manner may be determined the value of an assurance of £100 for 10 years on the longest of two lives, viz a man aged 46 and a woman aged 40; reckoning interest at 4 per cent, and the probabilities of life as observed in *Sweden*.

For, the value of such assurance will be equal to the difference between 33·031 and 30·067; that is, 2·964 will be the value in a *single* payment: and this sum divided by 8·251 (or unity added to the value of an annuity on the *longest* of the two lives for nine years) will give ·359 for the value of the same in *annual* payments during the *longest* of the given lives: or being divided by 7·329 (equal to unity added to the value of an annuity on the two *joint* lives for nine years) will give ·404 for the value of the same in *annual* payments during the existence of the *joint* lives.

SCHOLIUM.

§ 420. When we have to determine the value of a temporary assurance for a very short term, such as 1, 2, 3, &c years, it will be the easiest method to calculate the value of each year's expectation from the tables of mortality. For, the probability that a person of any given age will die in any particular year is a fraction whose denominator is the number of persons living at that age, and whose numerator is the number of persons that die within the given year; and which fraction, being multiplied by the present value of the given sum due at the end of the given year, will give the expectation of receiving such sum at the end of that year, provided the given life becomes extinct in that year: and the sum of these annual expectations for the 1st, 2nd, 3rd, &c years will be the value of the assurance for those periods respectively.

Example 1. What is the value of an assurance of £100, for *one* year, on the life of a woman aged 40; or, in other words, what is the present value of £100 to be received at the end of the year, provided such life be then extinct: interest being reckoned at 4 per cent, and the probabilities of life as observed in *Sweden?*

The probability that a woman of 40 will die within the *first* year is, by Table XIV, equal to $\frac{65}{4753}$; and the present value of £100 to be received at the end of a year is, by Table LVIII, equal to 96·154: these two quantities, multiplied together, will produce 1·321, or £1 : 6 : 5, for the value required.

In like manner it may be found that the probability of a woman, aged 40, dying within the *second* year is by the same Table equal to $\frac{75}{4733}$; and that the present value of £100 to be received at the end of two years is equal to 92·456: which quantities being multiplied together will produce 1·465 for the present value of £100 to be received at the end of the second year provided the given life becomes extinct in that year. And this value, added to the one above found, will give 2·786, or £2 : 15 : 9, for the value of the assurance for *two* years.

By a similar method of proceeding it will be found that $\frac{76}{4733}$ multiplied by 88·900 will give 1·428 for the present value of £100 to be received at the end of the *third* year on a similar contingency: and which value, being added to the sum of the two former ones, will give 4·214 for the value of the assurance for *three* years. And so on for the subsequent years.

Had it been required to determine the value of a similar assurance on the life of a man, aged 46, the expectations for the first, second, and third years would have been equal to $\frac{80}{3991}$ multiplied by 96·154, 92·456, and 88·900 respectively:* whence, those expectations would have come out equal to 1·927, 1·853, and

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* Because $\frac{80}{3991}$ is not only the probability that such life will die in the *first* year; but also the probability that it will die in the *second* year; and also in the *third* year: as may be seen by Table XIV.

1·782: and the value of the assurance, for one, two, and three years, would have been 1·927, 3·780, and 5·562 respectively.

The same observations will apply to assurances for one year on any *joint* lives. For, the probability that any two joint lives will fail within the first year is the difference between unity and the product of the probabilities that they shall both live to the end of the year: and which difference, being multiplied by the present value of the given sum due at the end of that year, will give the expectation of receiving such sum at that period provided either of the given lives be then extinct.

Example 2. What is the present value of an assurance of £100 for *one* year, on the joint lives of a man aged 46 and his wife aged 40: interest at 4 per cent, and the probabilities of life as observed in *Sweden?*

The probability that a man aged 46 will live to the end of the year is equal to $\frac{3911}{3991}$; and the probability that a woman aged 40 will live to the end of the same period is equal to $\frac{4668}{4733}$: these two fractions, therefore, being multiplied together and their product subtracted from unity, will leave ·0335; which being multiplied by 96·154 (or the present value of £1 due at the end of the year) will produce 3·221, or £3 : 4 : 5, for the value required.

§ 421. These examples will show the method of proceeding in all similar cases: and for the information of the reader I shall here subjoin a table of the sums demanded by the different Assurance Companies for the assurance of £100 for *one year* on a single life at the several ages therein mentioned: to which I shall add the fair value that *ought* to be given for the same, according to the probabilities of life as observed by *M. De Parcieux*,* and reckoning interest at 4 per cent.

Ages.	Northampton. 3 per cent.	De Parcieux. 4 per cent.
10	·890	·929
20	1·302	·900
30	1·661	1·037
40	2·030	1·049
50	2·753	1·431
60	3·906	2·983
70	6·184	5·289

From which it appears that the several Assurance Companies require, in most cases, half as much again as ought to be given; and in some cases nearly *double* the sum that should be given for the assurance. And though some compensation ought to be allowed for the expenses incurred in carrying on the business of the Office, as well as a proper remuneration for the services of those who conduct it; yet it is evident that these sums are greater than ought reasonably to be taken: particularly when it is considered that those who insure at *any** of the Offices, for a *term* of years only, have not much prospect of deriving any advantage from the profits of the concern.

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* The probabilities here alluded to are, in this particular case, taken from the Table of Observations given by Dr. Price in his *Obs. on Rev. Pay.* vol. ii. p. 456: because the *decrements* of life are there more correctly given; and, being on a more enlarged scale, are therefore more applicable to the present examples.

QUESTION XXX.

§ 422. To find the value of an *Assurance* of a given sum to be received on the decease of A, provided he dies *before* another given life B.†

SOLUTION.‡

Let O represent a life one year *older* than A; and Y a life one year *younger* than A. Add unity to the value of an annuity on the two joint lives OB, and multiply the sum by the number of persons living at the age of O; then divide this product by the amount of £1 in a year, and reserve the quotient. Multiply the value of an annuity on the two joint lives YB by the number of persons living at the age of Y: and, having subtracted the product from the reserved quotient, divide the remainder by the number of persons living at the age of A. Subtract this last quotient from the present value of £1 payable on the extinction of the two joint lives AB; and the remainder, multiplied by half the given sum, will be the value required. See § 231.

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* For, such persons do not (even at the Equitable Society) participate in the profits, unless a bonus happens to be declared during the term for which they are assured: which, in most ordinary cases, (if it occurs at all) is but a *partial* advantage.

† The present Question will be found of considerable utility in enabling us to determine the propriety and advantage of those Schemes which are formed with a view of providing *sums* of money to be paid to *Widows*, on the decease of their husbands. See what has been already said on the subject of *annuities*, applied in a similar manner, in the note in page 376.

‡ When the two lives are of the *same* age, the present value required is in all cases equal to the present value of half the sum payable on the extinction of the two joint lives AB. Thus, if the two lives, in the first example, were both aged 20, the present value required would be equal to 17·095: and if they had both been 40 years of age, the required value would have been equal to 22·634.

Example 1. What is the present value of £100 payable on the death of A aged 20, provided B aged 40 be then living: interest at $4\frac{1}{2}$ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on the two joint lives OB, aged 21 and 40, is, by the rule in page 365, equal to 12·520; which, being added to unity, and then multiplied by 806 (or the number of persons living against the age of 21, in Table III), will produce 10897·120; and this being divided by 1·045 will give 10427·866 for the reserved quotient. The value of an annuity on the two joint lives YB, aged 19 and 40, is, by the rule in page 365, equal to 12·575; which being multiplied by 821 (or the number of persons living at the age of 19, in Table III) will produce 10324·075: this being subtracted from 10427·866 (the reserved quotient) and the remainder divided by 814 (or the number of persons living at the age of 20) will give ·1275. Now, the value of £1 payable on the extinction of the joint lives AB, aged 20 and 40, is, by Question XXVII, equal to ·4167: therefore ·1275 being subtracted from this value will leave ·2892; which being multiplied by 50 will give 14·46, or £14 : 9 : 2, for the present value required.

Having thus found the present value of the given sum payable on the decease of A, provided B be then alive; we may easily determine the present value of the same sum payable on the decease of B, provided A be then alive. For, we have only in such case to deduct the value, found by the rule, from the value of the same sum payable on the extinction of the joint lives AB. Thus, the present value of £100 payable on the extinction of the joint lives AB is, by Question XXVII, equal to 41·67; whence, if we subtract 14·46 from such value, the difference, or 27·21, will be the present value of £100 payable on the decease of B, provided A be then alive.

These values are, in each case, the sums that ought to be given in a *single* payment: but, if we wish to determine the value of the same in *annual* payments, we must divide those sums by unity added to the value of an annuity on the two joint lives AB. Therefore 14·46 being divided by 13·545 will give 1·068, or £1 : 1 : 4, for the annual payments in the former case: and 27·21 divided by 13·545 will give 2·009, or £2 : 0 : 2, for the annual payments in the latter case.

Example 2. B aged 60 will, if he lives till the decease of A aged 25, be entitled to a legacy of £100: what is the value of his interest in such sum, taking the probabilities of living as at *Northampton*, and the rate of interest 5 per cent?

The value of an annuity on two joint lives OB, aged 26 and 60, is equal to 7·365; which being added to unity and multiplied by 4685 (or the number of persons living at the age of 26, as in Table XXV) will give 39190·025; and this, divided by 1·05, will give 37323·833 for the quotient, to be reserved. The value of an annuity on the two joint lives YB, aged 24 and 60, is equal to 7·399; which being multiplied by 4835 (or the number of persons living against the age of 24) will produce 35774·165: this, being subtracted from the reserved quotient, and the remainder divided by 4760 (or the number of persons living against the age of 25), will give ·32556. Now, the present value of £1 payable on the extinction of two joint lives, aged 25 and 60, is equal to ·60081: therefore ·32556, being subtracted from this value, will leave ·27525: which, being multiplied by 50, will give 13·762, or £13 : 15 : 3, for the present value of B's interest in the legacy.

If this sum be subtracted from 60·081 (or the present value of £100 payable on the extinction of the joint lives AB aged 25 and 60) the difference, or 46·319, will be the present value of the legacy payable on the death of B, provided A be then alive.

And either of these values, divided by unity added to the value of an annuity on the two joint lives, will show the annual payment which ought to be given by B or A respectively, in order to have the same assured to his heirs provided he dies before the other.

SCHOLIUM.

§ 423. The examples above given show the proper method of proceeding in all similar cases: and it may be here useful to remark that the values in Table LIII (which are adopted by *all* the Assurance Offices in London) have been computed from an incorrect rule given by Mr. Simpson,* and therefore cannot be depended upon when the life of A is very young, or when there is any considerable difference between the ages of the two lives. I shall however enter more at large on this singular fact, in Chapter XIV. See also what has been already said on this subject in the note in page 187.

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* In the *Supplement* to his Doctrine of Annuities, Prob. XXXII; and in his *Select Exercises*, Prob. XXXII. In using which rule, it should be observed that, when the reversion is a *sum* and not an *estate*, the value found by the rule must be divided by £1 increased by its interest for a year: as explained in page 162. Agreeably to this correction it will be found that Mr. Simpson's rule may be expressed by the formula $s \times \frac{1-\varrho BB}{2(1+\varrho)} \times \frac{\varepsilon}{e}$: which denotes half the value of an assurance of the given sum payable on the extinction of two joint lives of the same age with the oldest of the two lives, multiplied by a fraction whose numerator is the *expectation of the life* B, and whose denominator is the *expectation of the life* A.

This is the approximate value when B, or the life in expectation, is the oldest of the two lives. But if B be the youngest, this value must be subtracted from $s \times \frac{1-\varrho AB}{(1+\varrho)}$; and the difference will be the value in this case.

I would here observe that Mr. Dodson's formula (*Mat. Rep.* vol. iii. Ques. 23) is deduced from precisely the *same* series as Mr. Simpson's formula: nevertheless, they give *different* results when expounded numerically.

QUESTION XXXI.

§ 424. To find the value of a *Temporary assurance* of a given sum payable on the decease of A, provided he dies *before* another life B.

SOLUTION.*

Add 2 to the rate of interest; multiply this sum by the value of an annuity on the life B, and add unity to the product. Call this the *first* value.

Add 2 to the rate of interest; multiply this sum by the value of an annuity on a life older than B by the given term, add unity to the product, and then multiply this sum by the expectation of B's receiving £1 at the end of the term. Call this product the *second* value.

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* It may be necessary to observe that this rule is only an *approximation* to the true value, agreeably to the principles laid down in the note in page 194; and therefore must be always used, not only with caution but, with a due regard to the tables of observation employed. The correct value may be obtained by the help of the formula in Prob. XXVII, cor. 4; but, as that formula could not be conveniently expressed in words at length, I have preferred the one, above alluded to, for the illustration of this part of the subject.

Divide the probability that A will die before the end of the term, by the number of years, and multiply the quotient by half the given sum. Call this product the *third* value.

Subtract the *second* value from the *first*, and divide the remainder by the amount of £1 in a year: the quotient thence arising, being multiplied by the *third* value, will give the present value of the given sum required.

Example 1. What is the present value of £100 payable on the decease of A aged 7, provided that shall happen within 14 years, and provided another life B, aged 30, be then alive: interest at 3 per cent, and the probabilities of living as observed at *Northampton?*

The value of an annuity on the life B, aged 30, is, by Table XXVII, equal to 16·922; which being multiplied by 2·03 (or 2 added to ·03) will produce 34·35166; and this, being added to unity, will give 35·35166 for the *first* value.

The value of an annuity on a life 14 years older than B (that is, on a life aged 44) is equal to 13·929; which, being also multiplied by 2·03, will produce 28·27587. This, being added to unity, will give 29·27587; which being multiplied by ·50144 (or the expectation of B's receiving £1 at the end of

14 years*) will produce 14·68009 for the *second* value.

The probability that A will die before the end of the given term is equal to ·14599,† which being divided by 14 will give ·010428; and this quotient, multiplied by 50 (or half the given sum) will produce ·5214 for the *third* value.

The difference between the *first* and *second* value is 20·67157; which, being divided by 1·03, will give 20·0695. This quotient, multiplied by ·5214 will produce 10·464 for the value required.

This is the sum that ought to be given for the assurance in a *single* payment: but, if we wish to determine the value of the same in annual payments, we must divide this sum by 9·566 (or unity added to the value of an annuity on the two joint lives for 13 years); which will give 1·094 for the annual payment required.

Example 2. B aged 60 will, if he lives to the decease of A aged 25, be entitled to a legacy of £100, provided that event shall happen within 15 years: what is the value of his interest therein, reckoning the probabilities of life as observed at *Northampton*, and the rate of interest at 5 per cent?

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* The probability that B shall live to the end of 14 years is, by the rule in Question I, equal to $\frac{3326}{4385}$; and the present value of £1 certain to be received at the end of that period is, by Table LVIII, equal to ·6611: the product of these two quantities will give ·50144 for the expectation required.

† The probability that A shall *live* to the end of 14 years is, by the rule in Question I, equal to $\frac{5060}{5925}$; that is, equal to ·85401. This value, subtracted from unity, will give ·14599 for the probability that A shall *die* before the end of that period.

The value of an annuity on the life aged 60 is, by Table XXVII, equal to 8·392; which, being multiplied by 2·05 and added to unity, will make 18·2036 for the *first* value.

The value of an annuity on a life aged 75 is equal to 4·744; which being multiplied also by 2·05, and added to unity will make 10·7252: and this, being multiplied by ·19637 (or the expectation of B's receiving £1 at the end of 15 years*) will produce 2·1061 for the *second* value.

The probability that A will die before the end of 15 years is equal to ·23634†; which, being divided by 15, will give ·015756; and this quotient, multiplied by half the given sum, will produce ·7878 for the *third* value.

The difference between the *first* and *second* value is 16·0975; which, being divided by 1·05, will give 15·3310. This quotient, multiplied by ·7878, will produce 12·078 for the value required, in a *single* payment.

And this sum being divided by 7·592 (or unity added to the value of an annuity on the two joint lives for 14 years) would give 1·591 for the value of the same sum, if required, in *annual* payments.

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* The probability that B shall live to the end of 15 years is equal to $\frac{832}{2038}$, and the present value of £1 to be received at the end of that term is equal to ·48102: the product of these two quantities will give ·19637 for the expectation required.

† The probability that A will die in 15 years is equal to $\frac{3635}{4760}$ subtracted from unity; that is, equal to $\frac{1125}{4760}$, or ·23634.

SCHOLIUM.

§ 425. If the term, for which the assurance is made, happens to fall within the limits of equal decrements, of the life A, as found in the given table of observations, it is obvious (from the method of deduction in the note, in page 192, &c) that this rule will give the *exact* value. This is the case in the second example here given: for, by referring to Table XXV, it will be found that from the age of 25 to 40 the decrements of life are exactly equal: and consequently the rule is in this case strictly correct. Nevertheless, if the value of the same assurance be found by the help of the formula given in Prob. XXVII, cor. 4, it will come out equal to 12·139: and I can account for this difference in no other way, than by supposing there is some error in the tables of the values of the annuities; for, it is evident that the two results *ought* to be precisely the same.

QUESTION XXXII.

§ 426. To find the value of an *Assurance* of a given sum payable on the decease of A, provided he dies *after* another life B.

SOLUTION.

From the value of the assurance of the given sum payable on the decease of A, subtract the value of the same assurance payable on the decease of A, provided he dies *before* B: the difference will be the value required. See § 241.

Example 1. What is the present value of £100 payable on the decease of A aged 20, provided B aged 40 be then dead: reckoning interest at 4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of the assurance of £100, payable on the decease of A, is, by Question XXVII, equal to 24·107; and the value of the same sum, payable on the decease of A provided he dies before B, is, by Question XXX, equal to 14·460: consequently 9·647, or £9 : 12 : 11, will be the value required.

Having thus found the value of the given sum payable on the decease of A, provided B be then dead, we may easily determine the value of a similar sum payable on the decease of B, provided A be then extinct: for, we have only in such case to deduct the value, found by the rule, from the value of the same sum payable on the extinction of the longest of the two lives. Thus, the present value of £100 payable on the extinction of the longest of two lives aged 20 and 40 (at the rate of interest &c, above mentioned) is, by Question XXVII, equal to 16·748; from which if we subtract 9·647, as above found, the difference, or 7·101, will be the value of £100 payable on the decease of B, provided he dies after A.

These sums are the values which ought in each case to be given in a *single* payment: but, if we wish to determine the value of the same in *annual* payments till the claim *is determined*, we must divide the single payment, thus found, by unity added to the value of an annuity on the two joint lives. Or, if we wish to determine the value of the same in *annual* payments till the claim *becomes due*, we must divide the single payment by unity added to the value of an annuity on the single life, on which the assurance is made.* Thus, 9·647 being the value, in a *single* payment, of an assurance of £100 on the life A provided he dies after B; it follows that the value of the same assurance, in *annual* payments till the claim *is determined*, is equal to 9·647 divided by 13·545: that is, equal to ·712, or 14*s* : 3*d*. And that the value of the same assurance, in *annual* payments till the claim *becomes due*, is equal to 9·647 divided by 17·624: that is, equal to ·547, or 10*s* : 11*d*.

Example 2. What is the present value of £100 payable on the decease of A aged 25, provided he should die after B aged 60: interest at 5 per cent, and the probabilities of living as at *Northampton*?

The value of an assurance of £100 on the decease of A is, by the rule in Question XXVII, equal to

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[*] See page 353.

30·633; and the value of the same sum payable at the same period, provided B be then extinct, is, by Question XXX, equal to 13·762: consequently 16·871, or £16 : 17 : 5, will be the value required, in a *single* payment.

This value, being divided by 8·383 will give 2·013, or £2 : 0 : 3, for the *annual* payments till the claim *is determined:* or, being divided by 14·567, will give 1·158, or £1 : 3 : 2, for the *annual* payments till the claim *becomes due.*

QUESTION XXXIII.

§ 427. To find the value of a *Temporary assurance* of a given sum payable on the decease of A, provided he dies *after* another life B.

SOLUTION.

From the value of a temporary assurance of the given sum payable on the decease of A, subtract the value of a similar temporary assurance, payable on the decease of A provided he dies *before* B: the difference will be the value required.

Example. What is the present value of a legacy of £100 payable on the decease of A aged 25, provided he dies within 15 years, and provided also that another person B, now aged 60, be then dead: interest being reckoned at 5 per cent, and the probabilities of living as at *Northampton?*

The value of an assurance of £100 payable on the decease of a person aged 25, provided that should happen within 15 years, is, by the rule in Question XXIX, equal to 16·354*; and the value of a similar assurance, provided another person now aged 60 be then living, is, by Question XXXI, equal to 12·078: consequently this latter value subtracted from the former will leave 4·276, or £4 : 5 : 6, for the value of the assurance required.

SCHOLIUM.

§ 428. This rule will still be correct although the given term *exceed* the number of years between the age of B, and the oldest life in the table of observations: but, in such case, the value of the assurance payable on the decease of A provided he dies *before* B (being now for the *whole continuance* of the joint lives) must be found by Question XXX instead of Question XXXI: and the value thus found, being deducted from the temporary assurance on the life A, will give the value required†.

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* The value of the assurance on the *whole continuance* of the life is, by the rule in Question XXVII, equal to 30·633; and the value of the same assurance *deferred* for 15 years is, by the rule in Question XXVIII, equal to 14·279: consequently the difference between these two values, will be the value of the *temporary* assurance.

† Mr. Morgan has given a singular and troublesome rule for this case, in Dr. Price's *Obs. on Rev. Pay.* vol. 1, page 70, note i. It is neither simple nor *correct.*

Thus, if the term in the preceding example had been 40 years; then the value of the assurance on the life A for 40 years would, by the rule in Question XXIX, be equal to 27·682*; and the value of the assurance on the same life, provided B be alive at his decease, would, by Question XXX, be equal to 13·762; consequently, this latter value subtracted from the former would leave 13·920, or £13 : 18 : 5, for the value required, in a *single* payment. And 13·920 divided by 8·383 (or unity added to the value of an annuity on the two joint lives) will give 1·661 for the value of the same in *annual* payments till the *claim* is determined: or, being divided by 14·164 (or unity added to the value of an annuity on the life A for 39 years) will give ·983 for the same value in *annual* payments till the sum *becomes due*.

§ 429. Before I close the present chapter I shall insert the solution of a Question, which will be often found of considerable practical utility, not only to individuals, but likewise to those Societies whose business consist in granting Assurances on lives.

The reader must be already aware that if a person were to make an assurance at any of the Offices on his own life for a *single* year, and to repeat this at the end of every successive year to the utmost extremity of life, the annual payment (for such assurance) would be continually *increasing* till his death. But, if he made the assurance on the *whole continuance* of his life, and contracted with the Office to pay the value of such assurance by *equal* annual payments during his life (as is usually the case), it is evident that such annual payment ought to be *greater* than the premium required for an assurance for a single year at his *present age*, but *less* than the premium required for a similar assurance at the more *advanced periods* of life. Hence, it appears that if a person, who was originally assured for the whole term of his life, should be desirous (either through inability, or any other motive) of renouncing his claim upon the Office and of cancelling his policy, he ought to have some part of those annual payments returned to him: or, in other words, a compensation ought to be made him for that *excess* in the annual payments which he has been advancing to the Society. The object of the following question is to determine the amount of that remuneration.

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* The value of the assurance on the *whole continuance* of the life is, as in the preceding note, equal to 30·633; and the value of the same assurance *deferred* for 40 years is, by the rule in Question XXVIII, equal to 2·951: consequently the latter value subtracted from the former will be the value of the *temporary* assurance required.

QUESTION XXXIV.

§ 430. To find the sum that ought to be given to a person, who is assured for the whole term of his life, for a given sum, in order that he may *renounce his claim* thereto.

SOLUTION.

Subtract the equal annual payment, which he has been giving since the assurance commenced, from the equal annual payment which ought to be given for the assurance of the given sum on the life at its *present* age; multiply the remainder by the value of an annuity (increased by unity*) on the life at its present age: the product will be the sum required.

Example. A person *now* aged 50, who has been paying 21·790, or £21 : 15 : 10,† annually for the assurance of £1000 at his death, is desirous of discontinuing the same; what sum ought to be given to him, by the Office, as a compensation for so doing: interest being reckoned at 3 per cent, and the probabilities of living as at *Northampton?*‡

The annual premiums which ought to be given for the assurance of £1000 on a life aged 50 is, by Table LI, equal to 45·300; and the difference between this and 21·790 is equal to 23·510; which being multiplied by 13·436 (or unity added to the value of an annuity on a life aged 50) will produce 315·880, or £315 : 17 : 7, for the answer required.*

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* This supposes that the policy is cancelled immediately *before* the annual payment becomes due: but if immediately *after*, we must multiply the remainder, above alluded to, by the value of an annuity on the given life, without the addition of unity.

† This is the annual payment for the assurance of £1000 on a life aged 20, as appears by Table LI.

‡ The rate of interest and probabilities of life, in *such* computations, *ought* to be the *same* as those adopted by the Office, at which the policy is effected.

SCHOLIUM.

§ 431. If the sum, which is to be received at the death of the assured, has been increased by any addition, or bonus, (as will oftentimes be the case in the *Equitable Society*) we must take the difference between the annual payment, that ought to be given for the assurance of the given sum *together with its bonus*, and the original annual payment above mentioned: and the remainder multiplied, as before, by the value of an annuity (increased by unity) on the life at its present age, will give the sum required.†

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* This case may be stated in another manner, as follows. The Society may be considered as indebted to the assured in the present value of an assurance of £1000 on a life aged 50; which is equal to 608·66, or £608 : 13 : 2. And the assured may be considered as owing to the Society the present value of all the annual payments of £21 : 15 : 10, during the remainder of his life; the first of which payments is supposed to be made immediately: therefore the value of all those payments will be equal to 21·790 multiplied by 13·436; which produces 292·78, or £292 : 15 : 7. Consequently the interest of the assured in his policy will be equal to the *difference* between £608 : 13 : 2, and £292 : 15 : 7: that is, equal to £315 : 17 : 7, as found by the example in the text.

† Or, we may add the value of an assurance of the *additional* sum on the given life, to the value found by the preceding solution; which will give the same results.

Example. Suppose that the several additions made to the policy, alluded to in the preceding example, amount to £500; in which case, the executors of the assured would be entitled to £1500 on his death: what sum ought now to be given him for renouncing his claim upon the Society; the interest, &c, being as in the preceding example?

The annual premium which ought to be given for the assurance of £1500 on a life aged 50 will be 67·950; and the difference between this and 21·790 is equal to 46·160: which being multiplied by 13·436 will produce 620·206, or £620 : 4 : 1, for the answer required.

The same result would have been obtained by adding £304 : 6 : 6 (or the value of an assurance of the additional £500 on the given life) to £315 : 17 : 7, as found by the preceding example.

§ 432. These two examples will show the method of proceeding in all similar cases, whether the assurance depends upon a single life, upon any joint lives, or on any other contingency. It will also serve to show the amount of the *debts* owing by a Company, whose business consists in making Assurances on lives: since these are the sums which would be required to cancel the respective claims on the Society, and may consequently be fairly considered as money owing by them. These debts, therefore, being deducted from the amount of capital in hand, will leave the net *surplus* stock belonging to the Society: and it is this net surplus stock *alone* that can be considered as the true *profits* of the concern, and as the *only* fund from which any divisions ought to be made amongst the different members; either by way of interest, dividend, or bonus. A society that is not guided by some principle of this kind must inevitably terminate in disgrace and ruin.

CHAPTER XIII.

On Schemes for providing Annuities for the Benefit of Old Age, and of Widows.

1. *For Old Age.*

§ 433. THE rule given for the solution of Question VI, in the preceding Chapter, is of considerable practical use; since, by means of it, we are enabled to determine the efficacy and propriety of those schemes and establishments which are proposed with a view of providing *Annuities for Old Age.* For, having found the present value of an annuity of £1 per annum on any given life or lives, to commence at the end of any number of years from the present time, we may easily find the present value of any *other* annuity (either in a *single* payment, or in *annual* payments, or in *both*) by multiplying the present values, thus deduced, by that other annuity; agreeably to the principles already laid down in the note in page 362. And in like manner, having found the present value above alluded to, we may readily determine the annuity that ought to be given, at the end of any period, for a given sum paid down immediately; or, for a given sum, *part* of which is paid immediately, and the *remainder* by annual payments till the end of the given term.

Thus, it appears, by the second example in page 378, that the present value of an annuity of *one pound* per annum on the life of a female aged 40, to commence at the end of 10 years, is (reckoning interest at 4 per cent, and the probabilities of life as observed in *Sweden*) equal to 6·926, or £6 : 18 : 6 : consequently an annuity of £44 per annum to commence at the same period would be equal to 304·744, or £304 : 14 : 11, in a *single present* payment. And this latter sum divided by 7·900 (or unity added to the value of an annuity on the given life for 9 years) will give 38·575, or £38 : 11 : 6, for the value of the same annuity in *annual* payments: the first of such payments to be made immediately, and the remaining ones at the beginning of every subsequent year, provided the given life is then in being. Or, if £200 (part of the £304 : 14 : 11) be paid down *immediately*; then the *annual* payments which ought in this case to be made, in order to supply the deficiency, will be equal to the remaining sum of 104·744 divided by 7·900:* that is, equal to 13·259, or £13 : 5 : 2.

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* If the first of these annual payments is not made till the *end* of the first year, then the sum here alluded to must be divided by 7·475 (or the value of an annuity on the given life for 10 years) which will make the annual payments equal to 14·013, or £14 : 0 : 3 : and, in this case, the last annual payment will be made at the same time that the first payment of the annuity becomes due. In the cases mentioned by Dr. Price (*Obs. on Rev. Pay.* vol. i. p. 141) and to which I shall presently allude, the payments were made *half*-yearly; and the first of those half-yearly payments was paid down with the Admission money: but he does not notice this fact, although it makes some difference in the results.

In like manner if we wish to determine the *annuity* that ought to be granted on such life at the end of 10 years, for £73 : 10 : 0, paid down *immediately*, and £6 : 14 : 0 in *annual* payments, we must proceed as follows. Multiply 6·700 (or the amount of the annual payment) by 7·900 (or unity added to the value of annuity on the given life for 9 years)*, which will produce 52·930, or £52 : 18 : 7, for the total present value of such annual payments; and which being added to the admission money, or £73 : 10 : 0, will make the whole value, given for the annuity, equal to 126·430, or £126 : 8 : 7. Therefore this quantity, being divided by 6·926 (agreeably to the principles laid down in the note in page 363), will give 18·254, or £18 : 5 : 0, for the amount of the annuity required.

§ 434. This example will show the method of proceeding in all similar cases; and I have taken this one in particular, because it is the one assumed by Dr. Price (*Obs. on Rev. Pay.* vol. i. p. 137) to expose the futility and iniquity of those Schemes that were published by the several societies instituted about the year 1770: who, by holding out a false lure to the public, took in the unwary, and entailed misery and distress on the unfortunate adventurers.† Happily his efforts were crowned with success, and none of those public societies now exist to disgrace the present age. But although he has sufficiently demonstrated that those bubbles could not possibly comply with the terms

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* If the first annual payment is not to be made till the *end* of the year, we must multiply 6·700 by 7·475 (or the value of an annuity on the given life for 10 years): see the preceding note.

† The following are the terms upon which some of those So-

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cieties granted annuities to members, aged 40, for what might happen to remain of their lives after 50 years of age.

	Admission Money.	*Annual Payment.*	*Annuity granted.*
	£ s.	£ s.	£ s.
The Laudable Society	73 : 10	6 : 14	44 : 0
The London Annuitants	25 : 0	10 : 0	44 : 0
The Equitable Society of Ann^{s}. .	32 : 0	13 : 0	44 : 0
The London Union Society	37 : 0	7 : 0	54 : 12
The Amicable Society of Ann^{s}. .	28 : 16	6 : 0	26 : 0
The Provident Society	31 : 10	8 : 8	25 : 0

The annual payment was usually paid *half* yearly; and the *first* half yearly payment was made at the time of admission. It will be readily seen (from the example given in the text) that there was no probability that the Societies would ever be able to *continue* these enormous annuities to *all* the members. The truth is, that (even at that day) they were styled *impositions on the public proceeding from ignorance, and supported by credulity and folly.* "But (as Dr. Price justly continues) this is too gentle a censure. "There is reason to believe that *worse* principles have contributed "to their rise and support. The present members, consisting "chiefly of persons in the more advanced ages who have been ad-"mitted on the easiest terms, believe that the schemes they are "supporting will last their time and that they will be gainers. "And, as to the injury that may be done to their successors or to "younger members, it is at a distance, and they care little about it. "Agreeably to this principle the founders of these societies begin "so low as not to require perhaps a *fourth* or a *fifth* of the values "of the annuities they promise. Afterwards they advance gra-"dually, just as if they imagined that the value of the annuities held out to the public, yet he has not shown that they were *so* deficient and absurd as they will appear to be, when compared with the values deduced as above from the real probabilities of life.* The present examples will be sufficient to enable any one to examine the accuracy and sufficiency of similar proposals: and I shall not detain the reader with any further comments on this subject, as it is probable that a similar circumstance will not soon again occur to call forth the censure of every honest member of society. The public are now better acquainted with the method of calculating the values that ought to be given in such cases, and are not likely to become again in *this* manner the dupes of any artful impostor. Their danger now seems to lie in the *opposite* extreme.

“ was nothing determinate, but increased with every increase of the society. But, as no ignorance can believe this, the true design appears to be, to form soon as large a society as possible by leading the unwary to endeavour to be *foremost* in their applications, lest the advantage of getting in on the easiest terms should be lost. It is well known that these arts have succeeded wonderfully; and that, in consequence of them, these societies now consist of persons who for the *same* annuities make higher or lower payments, according to the time when they have been admitted; and the generality of whom, therefore, must know that either more than the values have been required of the members last admitted; or, if not, that they are themselves expecting considerable annuities for which they have given no valuable consideration, and which, if paid, must be *stolen* from the pockets of some of their fellow-members.”

* Dr. Price formed his calculations from the values of annuities deduced from the hypothesis of *M. De Moivre.*

I do not, however, by these observations at all intend to discourage institutions or schemes for providing annuities for old age. On the contrary, I am fully persuaded that a society or office, that would proceed upon an efficient and *liberal* plan, might be of essential advantage to the state and highly beneficial to the public. Many persons, particularly in the inferior stations of life, would in such case be induced to lay by (during the period of youth and vigour) many small sums, which are now squandered in riot and dissipation; and by endeavouring to get a little money beforehand would acquire habits of industry, and be probably enabled thus to make provision for themselves in the more advanced periods of life, when they will be incapable of labour. Thereby rendering themselves not only more independent and valuable members of society, but also obviating the necessity of their applying to the parish for relief. Dr. Price justly observes that "the lower orders of mankind are objects "of particular compassion, when rendered incapable, "by accident, sickness, or age, of earning their sub"sistence. This has given rise to many very useful "societies among them for granting relief to one an"other, out of little funds supplied by weekly contri"butions." It is much to be feared, however, that many of these establishments are formed on such vague and inefficient plans, without any regard to the true principles of calculation, that they are not always entitled to our unqualified approbation. When thus erroneously founded, and perhaps at the same time badly and carelessly conducted, they only serve to increase the misery which they were intended to remove.

§ 435. Under these circumstances it is to be regretted that the Legislature has not adopted some efficient measures towards assisting such persons, in the humbler stations of life, as might be desirous of employing their money in this laudable manner. Two attempts have been made to obtain an Act for this purpose: but, though it has each time passed the House of Commons, it has been as regularly thrown out by the House of Lords. I cannot here, however, omit the opportunity of inserting the pertinent remarks, on this subject, of a very able writer*, which will serve to show the objects that those bills had in view.

'To make such a provision for one's old age, is so 'natural a piece of prudence, that it seems wonderful 'at first sight that it should not be generally practised 'by the labouring poor above described, as it is almost 'universally by persons engaged in the higher paths 'of industry, such as the several branches of com-

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* Mr. Baron Maseres, in a pamphlet which he published, entitled *A Proposal for establishing Life Annuities in Parishes for the Benefit of the Industrious Poor:* London, 1772. The learned author has lately favoured me with a copy of that pamphlet, with an obliging offer that I might make what use of it I pleased in my intended publication: and I lament that the limits of the present treatise will not enable me to enter upon the subject more than by an insertion of the above extract from that valuable work.

'merce, and the practice of the liberal professions: 'nor can their negligence in this respect be accounted 'for in any other way so naturally as by ascribing it to 'their wanting opportunities of employing the money, 'they might save, in some safe and easy method that 'would procure them a suitable advantage from it in 'the latter periods of their lives. They know, for 'the most part, but little of the public funds; and 'when it happens that they are acquainted with them, 'the smallness of the sums they would be intitled to 'receive (as the interest of the money they could afford 'to lay out in them) is no encouragement to dispose 'of it in that way. What inducement, for instance, 'can it be to a man, who has saved ten pounds out of 'his year's wages, to invest it in the purchase of three 'per cent Bank annuities, to consider that it will pro'duce him about six or seven shillings a year? It is 'but the wages of three days labour. And if they 'lend their money to tradesmen of their acquaintance, 'as they sometimes do, it happens not unfrequently 'that their creditor becomes a bankrupt; and the 'money, they had trusted him with, is lost for ever; 'which discourages others of them from saving their 'money at all, and makes them resolve to spend it in 'the enjoyment of present pleasure. But, if they saw 'an easy method of employing the money they could 'spare, in such a manner as would procure them a 'considerable income in return for it at some future 'period of their lives, without any such hazard of 'losing it by another man's folly or misfortune, it is

‘ probable they would frequently embrace it: and thus ‘ a diminution of the poor’s rate on the estates of the ‘ rich, an increase of present industry and sobriety in ‘ the poor, and a more independent and comfortable ‘ support of them in their old age than they can other‘ wise expect, would be the happy consequences of ‘ such an establishment. Now this, I conceive, might ‘ be effected in the following plain and easy method.

‘ 1° Let the churchwardens and overseers of every ‘ parish be impowered by Act of Parliament to grant ‘ life-annuities to such of the inhabitants of the parish ‘ as shall be inclined to purchase them, to commence ‘ at the end of one, two, or three years (or such other ‘ future period of time as the purchaser shall choose) ‘ and to be paid out of the poor's rate of the parish, ‘ so that the lands and other property in the parish ‘ that is chargeable to the poor’s rate, shall be answer‘ able for the payment of these annuities. This cir‘ cumstance would give these annuities great credit ‘ with the poor inhabitants, by setting before them so ‘ solid and ample a security for the regular payment ‘ of them.

‘ 2° Let the annuities, thus granted to the poor in‘ habitants, be such as arise from a supposition that ‘ the interest of money is three per cent; or some higher ‘ rate of interest, if the churchwardens or overseers of ‘ the poor shall think fit to make use of such higher ‘ interest.

‘ 3° But, at the rate of three per cent, the purchaser ‘ should have a right to an annuity, and the church-

‘ wardens and overseers of the poor should be compellable to grant it.

‘ 4° No annuity, depending upon one life, should ‘ exceed 20l. a year.

‘ 5° No less sum than *5l.* should be allowed to be ‘ employed in the purchase of an annuity. This is to ‘ avoid intricacy and multiplicity of accounts.

‘ 6° An exact register of these grants should be ‘ kept by the churchwardens and overseers of the poor ‘ in proper books for the purpose, in which the grants ‘ should be copied exactly, and the copy of each grant ‘ subscribed by the person to whom it is granted : or, ‘ if he cannot write, marked with his mark, and subscribed by two other persons as witnesses. And this ‘ copy in the register-book of the parish should be ‘ good evidence of the purchaser's right to the annuity, ‘ in case the original deed of grant to the purchaser ‘ (which was delivered to him at the time of the purchase) should afterwards be lost.

‘ 7° The money, thus paid to the churchwardens and ‘ overseers of the poor for the purchase of life-annuities, ‘ should be employed in the purchase of three per ‘ cent Bank annuities, in the joint names of all the ‘ churchwardens and overseers; and by them transferred ‘ at the expiration of their offices to their successors, ‘ and so on to the next successors for ever, so as to ‘ be always the legal property of the churchwardens ‘ and overseers of the poor for the time being, in trust ‘ for the persons who should be entitled to the several ‘ life-annuities, granted in the manner above men-

‘ tioned; and the interest of this money should be ‘ received every half year, and invested in the purchase ‘ of more principal annually, so as to make a perpetual ‘ fund for the payment of these life-annuities. And ‘ when any annuity became due, the churchwardens ‘ and overseers of the poor should pay it out of this ‘ fund, and should sell a sufficient part of the principal ‘ for that purpose, where the interest was not sufficient ‘ for the purpose, as would generally be the case: ‘ and the deficiencies (if any were) of both principal ‘ and interest should be made good out of the poor's ‘ rate assessed upon the parish. But this could hardly ‘ ever happen, if the annuities granted to the purchasers were such as would be proportional to the ‘ money contributed, upon a supposition that the interest of money was only three per cent; because that ‘ is a lower interest than that which the parish would ‘ receive from the government for the same money, ‘ by investing it in the three per cent Bank annuities, ‘ as that stock is now twelve per cent under par,* and ‘ is not likely soon to rise to par again. So that the ‘ landholders of the parish, and the owners of other ‘ rateable property in it, need be under very little apprehension of having their estates exorbitantly burthened by a great increase of the poor's rate, in order ‘ to make good the payment of these annuities. On ‘ the contrary, they would be gainers by this institution, ‘ as was observed above, since many of the poor who

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* It is now (1810) above 30 per cent under par.

'must otherwise, in their old age, come to be a bur'then upon the parish, would now be maintained, in 'part at least, by annuities paid to them out of a fund 'of their own raising.'

Agreeably to this plan, a bill* was brought into the House of Commons in the year 1773 by Mr. William Dowdeswell, and passed, on a division after a debate, by a majority of two to one of all the members present: but it was rejected by the House of Lords in consequence of the opposition of Lord Cambden, who conceived that the measure might be ultimately injurious to the *landed interest;* since the value of future leases might be affected by an increase of the poor's rate to make good any deficiency arising from the failure or defection of the scheme. A bill of a similar nature, however, with tables computed by Dr. Price,† was introduced in the year 1789; but shared a similar fate.

2. *For Widows.*

§ 436. The rules, by which Question XIII and XVIII, in the preceding Chapter, are solved, are very useful in enabling us to determine the advantage and propriety of those schemes which are instituted with a view to provide *Annuities for Widows**. For, having found the present value of a reversionary annuity, or a deferred reversionary annuity, of £1 per annum on any given life after any other given life, we may readily find the present value of any other annuity (either in a *single* payment, or in *annual* payments, or in *both*) by multiplying the present value, thus obtained, by that other annuity. And in like manner, having found the present value above alluded to, we may easily determine the annuity which ought to be granted, to the life in reversion, for a given sum paid down immediately; or, for a given sum, *part* of which is paid down immediately, and the *remainder* by annual payments during the existence of the life in possession. Or, according to any other plan which may be proposed.

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* A copy of this bill, together with the tables that were computed for it, are inserted by Mr. Baron Maseres in his *Doctrine of Life Annuities.*

† These Tables are inserted in Dr. Price's *Obs. on Rev. Pay.* vol. ii. page 473.

Thus, let the scheme of a society, for granting annuities to widows, be such that, if a member lives *one* year after admission, his widow shall be entitled to a life annuity of £20: if *seven* years, to £10 more (or £30 in the whole): if *fifteen* years, to £10 more (or £40 in the whole). What *ought* to be the annual payments for members aged 30, 40, and 50 years respectively, and supposing them of the same ages as their wives: interest being reckoned at 4 per cent, and the probabilities of living as observed in *Sweden* amongst males and females respectively; and also as observed at *Northampton* and at *London?*

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* These observations relate to the method of determining the best mode of providing *Annuities* for Widows: but those inquiries which relate to the best mode of providing for the payment of any given *Sum* to a widow, on the death of her husband, are answered by Question XXX, in the preceding Chapter.

By proceeding according to the rules laid down in the Scholium in page 394*, we shall find that the annual payments which ought to be made by the members, of the respective ages above mentioned, will (according to the several tables of observations made use of) be as follow:

Ages.		*Sweden.*	*Northampton.*	*London.*
30	=	6·90	7·66	8·52
40	=	7·89	8·09	9·06
50	=	8·50	8·40	9·51

I have selected this scheme as being that on which the *London Annuity Society for the Benefit of Widows* set out in 1765. The *Laudable Society*, which was formed on nearly a similar plan, had been established in the year 1761. In each of these, the *annual* contribution of every member was five guineas only, payable half-yearly; for which payment, his widow was entitled to the annuities stated in the scheme above mentioned, according to the conditions there expressed. Nothing, however, can more fully show the inadequacy of the means for carrying into effect the intentions of those Societies than the examples above given. For it will be seen that, on the supposition that the members are of the *same ages* as their wives, the Societies received, on an average, little more than *three fifths* of the true value of the annuities: but, on the supposition that they are, one with another, *ten years older* than their wives, it will be found that they received only *one half* of the true value of such annuities. The consequences of such inequitable measures were highly injurious. The more early annuitants enjoyed the *full* sums, according to the conditions of the plan: but, after pursuing this pernicious course for some years, the Directors at length listened to the voice of reason; and, in consequence of the repeated warnings that were given them, found it necessary to adopt one of the two following plans in order to maintain the permanency and security of the establishment:—to *increase* the premiums, or *diminish* the annuity. The *London Annuity Society*, at an early period, adopted the former proposal, and thus preserved its honour and its credit: but the *Laudable Society*, although it repeatedly reduced the annuities, had been too long struggling with the errors of its original establishment to enable it to derive much permanent advantage from this procrastinated remedy. "If circumstances therefore should still continue unfavourable, the next measure must be the dissolution of the Society, and a division of the remaining capital among the annuitants and surviving members, in proportion to their respective interests in the funds of the Society*."

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* For example: the value of a reversionary annuity on the life of a female aged 30 after the life of a man aged also 30, and deferred for *one* year provided *both* the lives continue so long, is (by the rule in the Scholium, according to the *Sweden* observations) equal to 3·108: consequently the present value of an annuity of £20, under the same circumstances, is 62·160. In like manner, the present value of £10 per annum on the same life, deferred for *seven* years provided *both* the lives continue so long, is equal to 21·680. And by the same rule the present value of £10 per annum, deferred for *fifteen* years, is under the like circumstances equal to 12·566.

These values, being added together, are equal to 96·404; or the value of the expectation, described in the scheme, in a *single* present payment: and which being divided by 13·965 (or unity added to the value of an annuity on the two joint lives) will give 6·903 for the value of the same expectation in *annual* payments, as stated in the text. In like manner may be found the answer to all Questions of a similar kind.

§ 437. Such will be the final issue of every scheme that is not founded on correct observations and on mathematical principles: and though my remarks have been confined to the two establishments above mentioned, they will, I fear, apply with too much justice to several others of a similar nature. "There are in this kingdom many institutions for the benefit of widows besides the two on which I have now remarked; and in general, as far as I have had any information respecting them, they are founded on plans equally inadequate, having been formed just as fancy has dictated, without any knowledge of the principles on which the values of reversionary annuities ought to be calculated. The motives which influence the contrivers of these institutions,

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* See the history of these two Societies, brought down to the present period, in Dr Price's *Obs. on Rev. Pay.* vol. i. pages 72—104.

" may be *laudable*; but they *ought* (I think) to have " informed themselves better."—" The longer such " schemes are carried on, the more mischief they " must produce. 'Tis vain to form such establish- " ments with the expectation of seeing their fate de- " termined *soon* by experience. If not more extrava- " gant than any ignorance can well make them, they " *will* go on prosperously for 20 or 30 years; and if " at all tolerable, they *may* support themselves for 40 " or 50 years, and at last end in distress and ruin. " All inadequate schemes lay the foundation of *present* " relief on *future* calamity; and afford assistance to a " *few* by disappointing and distressing *multitudes*.*"

The very learned and able writer, from whose work these quotations are made, employed his great abilities in detecting the pernicious tendency and iniquity of the several schemes above alluded to. His remarks on this subject are invaluable, and will always be consulted with advantage. He has not only shown how far the various societies, then in existence, erred from the true line of equity and propriety (predicting, therefrom, their incapacity and ruin); but has likewise pointed out some of the *best* schemes for providing annuities for widows—such as might be " durable, and " at the same time easy and encouraging." As I cannot add much to the observations of so intelligent an author I must refer the reader to that work, for any further information on this subject.

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* Dr. Price's *Obs. on Rev. Pay.* vol. i. Chapter II.

CHAPTER XIV.

An Account of the several Companies, established in London, for Life Assurances, &c.

The Amicable Society.

§ 438. THE first establishment of this kind, in point of time, is the *Amicable Society* for a perpetual Assurance office; which was founded by a charter of Queen Anne in 1706, at a period when the subject of Life Assurances was but little understood. Its original plan was consequently in many instances exceedingly defective, absurd, and inequitable: and, though it has since undergone several partial reforms, it is still liable to many objections. It was limited by its first charter to the disposal of shares or numbers, not exceeding 2000, held by *single* lives; and entitling to claims when they drop. For each of these shares every new purchaser paid down at entrance £7 : 10 : 0, besides £1 : 11 : 0, as the first quarterly payment of £6 : 4 : 0 per annum to be continued during his life. An annual dividend of £1 : 4 : 0 for each share was allowed to every purchaser, out of the profits of the society; which reduced the annual payment, on each share, to £5. No one person was allowed to purchase more than *three* shares; nor were any admitted to be purchasers whose ages were below 12 or above 45: and all between those ages were admitted on the *same* terms. Every person admitted became a member: and the net annual income, *arising from all the annual payments**, was equally divided among the nominees of such members as happened to die within the year. In 1757 the society engaged that this dividend, though it might be more, should not be *less* on each share than £125; and in 1770 they guarantied that it should not be less than £150.

Such are the outlines of the plan upon which the business of this society was *formerly* conducted. Its absurdity and injustice are manifest in its two most prominent features: 1° in regulating the dividends, among the *nominees*, by the number of members that die within the year; whereby that benefit is made to depend not on the value of the contribution, but on a *contingency:* 2° in requiring, from all the members, the *same* payments without regarding their differences of age. The utility of this institution however (defective as it is) has been virtually acknowledged, and the plan thereof sanctioned, by the grant of three subsequent charters: one in 1730, providing for the better regulation of the society; the next in 1790, enabling it to enlarge the number of its members to 4000;* and the last in 1807, whereby the society is enabled to render itself more extensively useful, and whereby it has obtained several other powers for the better regulation and conduct of the business of the corporation.

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* Making £10,000 when the society was full; being £5 net income on each of the 2000 shares. The *admittance money* of £7 : 10 : 0, together with such profit, interest or improvement, as the Society might be able to make, were carried to a *separate* account. In 1757 these savings had accumulated to £25,300 three per cent stock; which in 1770 had increased to £33,300. Out of this stock, the sum of £4000 Consols and £11,000 Reduced was in 1770 appropriated and set apart, together with all future savings, as a fund for augmenting the claims to the full sum of £150, whenever they fell short of that amount under the dividend directed by the charter. In 1780 the stock of the Society had been increased to £51,300; and in 1790, to £64,300 three per cent annuities, together with some property in houses. Out of this amount, the sum of £30,000 stock was appropriated in aid of the fund above mentioned: but the remainder cannot be disposed of without the consent of two successive General Courts.

By this new charter the society is enabled to grant *twelve* shares on each life at any age between 8 and 67. It is also empowered to proportion the premiums on admission, and the annual contributions for each share, to the age and other circumstances of the parties on whose lives the Assurance is made; with further power to grant (when applied for by the parties) Annuities in lieu of the amount insured, or in lieu of policies which it may be desirable no longer to continue. The *whole* of the annual contributions for every share granted previous to this charter, and *seven-eighths* of the contributions on shares subsequently granted, are to be divided every year amongst the claimants by deaths which happen in that year: and the corporation now engages that each share (although it may be more) shall not produce *less* than £180.*

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* When the Society, in 1770, guarantied that the dividend on each share should not be less than £150, the number of subscribers increased very considerably: the natural consequence of which was, (in a Society so constituted) that, for the first 10 or 15 years, the dividends rose from £150 to £200; and in some years even to £300. A similar rise was produced from the increase of the number of shares, by the charter of 1790: but these effects were *temporary* only, and necessarily ceased with the cause that produced them. They were sufficient however to *prove* to the society the absurdity of their plan, had they been disposed to profit by experience. Of late years, the dividend has been gradually decreasing; but it is *now* fixed that it shall not be less than £180.

The several charters provide that the joint stock,

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* The following are some of the *Rates*, on which persons are now admitted to become shareholders, according to the ages of the lives assured.

Ages.	*Premium on Admission.* £	s.	d.	*Annual Payments.* £	s.	d.
15	1	11	6	3	10	6
20	1	14	6	4	0	6
25	1	17	0	4	9	0
30	1	19	6	4	19	0
35	2	2	6	5	11	6
40	2	5	0	6	7	6
45	2	9	0	7	7	0
50	2	13	0	8	12	0
55	2	16	6	10	4	0
60	3	1	6	12	6	0
65	3	6	6	15	6	0

The annual payments are to be made *half* yearly; and the first half yearly payment is to be made at the time of admission.

raised by the accumulation of the reserve found necessary for the security of this guarantee, is to be improved for the benefit of all the members, according to their respective interests: and no division of this capital is to be made, otherwise than in the payment of claims on death, without the consent of two successive General Courts of the members of the corporation. And the Society is now enabled to effect assurances *for any number of years*, or on the *joint* continuance of lives, or on *survivorships*, or on *any other contingency of life.*[*]

The Royal Exchange, and London Assurance Companies.

§ 439. The *Royal Exchange Assurance* Company was formed about the time of the South Sea Bubble; and is one of the few projects which has survived that troublesome period,[†] and preserved its credit to the present day. It was established by charter dated June 22, 1720, under the name of *The Royal Exchange*

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[*] No table has yet been published of the terms on which *these* Assurances are effected.

[†] Out of above *two hundred* visionary schemes, that were formed and *carried into effect* at that time, only *four* exist at present! viz. the Royal Exchange Assurance Company, the London Assurance Company, the York Buildings Company, and the English Copper Company. See Macpherson's *Annals of Commerce*, vol. iii, page 99.

Assurance, for insuring ships and goods at sea, or going to sea, and for lending money on bottomry. This charter was granted by virtue of an Act of Parliament (6 Geo. I, c. 18): but in the following year (April 29, 1721) the king (by virtue of his prerogative) granted them another charter for the Assurance of *Lives* and against casualties and accidents by fire, under the name of *The Royal Exchange Assurance of Houses and Goods from Fire.*

The *London Assurance* Company had a similar origin, and was incorporated in consequence of the Act above mentioned. It is also empowered to grant Assurances on Lives.

In addition however to the powers and privileges, which are common to both these Companies, the *Royal Exchange* was further authorised and enabled (by Act 33 Geo. III, c. 14) to sell and purchase *Annuities* either immediate or reversionary, under certain restrictions: and for the purposes of that act only they are (over and above the names by which they were before styled) to assume the name of *The Royal Exchange Assurance Annuity Company.* In order to render full security to the *Annuitants* under this act, it is provided that the money received for the annuity shall (as often as it shall amount to £1000) be laid out in government security; the interest or dividends on which, the Corporation may from time to time receive: but they are prohibited from touching any part of the *principal* till the extinction of the life on which the annuity is granted. And for the sake of preventing any fraud in this respect, it is enacted that the time and place of the death and burial of each annuitant shall be inserted twice at least in the London Gazette: and that a certificate of such death or burial, with an account of the annuity depending, shall be left at the Office in London; to be inspected, without fee or reward, by any person interested in any annuity granted by virtue of that act. And it is likewise further enacted, that no agreement for the *selling* or *purchasing* of any annuity shall be valid, unless at the price stated in a Table which is directed to be hung up in some conspicuous place in the Office; and which table shall be remaining in the office at the time of granting the annuity.

It is difficult to discover upon what principles this Corporation has deduced its table of the *Rates* of Life Annuities.* Those rates have certainly not been deduced from any observations hitherto *published*; neither do they agree with any *probable* rate of human mortality: but seem to have been formed at random, without any regard to the true principles of the science. The values for all the lives *under* 50 years of age are much too *low*, even when compared with *M. De Parcieux's* Observations, at 4 per cent: but, *above* that age, they err *considerably more* from the true value, in the *contrary* extreme. If however their rates be compared with those proposed by Government (as explained at the end of this chapter), they will be found *altogether too high:* and few persons, who are enabled to accept the proposals there alluded to, will be disposed to sacrifice so considerable a difference by *purchasing* an annuity at the Office of the Royal Exchange Company: or indeed at any other Office whose terms have yet been published.

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* Their rates of Life *Assurances,* as well as those of the *London Assurance* Company, are the same as those adopted by *all* the other Offices: of which I shall speak hereafter.

Equitable Society.

§ 440. These were the only three Companies for granting Life Assurances till the establishment of the *Equitable Society* in 1762;* which was formed in consequence of Mr. Simpson's lectures, recommending such an institution. It appears likewise that Mr. Dodson was active in recommending the plan of this society; and composed some tables for its use. It is truly deserving the name which it has assumed: it being certainly one of the most *Equitable,* as well as the most important, of all the societies that have ever yet been formed for the purpose of granting Assurances on lives.

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* Several Societies however had been formed, and still exist, for the purpose of assuring the lives of a *particular class* of the community; such as Naval and Military men, Clergymen, Schoolmasters and others. But it is not my intention to enter into any inquiry on the principles whereon *these* Societies have been established.

By its constitution, the assured are *mutual* assurers one to the other; and participate equally* in all the profits and advantages of the concern. These profits are estimated from certain periodical† valuations of all the claims upon the society, compared with its present assets.‡ *One third* of the clear *surplus* stock is then set apart as a fund for the permanency and future security of the concern.§ But, the remaining *two thirds* are divided amongst the assured; and the amount of each person's share (estimated in proportion to the time that he has been a member) is added to his policy: so that his representatives will, at his decease, probably receive *considerably more* than the original sum insured. That this has hitherto been the case will appear from the following statement.

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* This applies only to those persons who are assured for the *whole continuance* of their lives: that *right* has not yet been extended to those who are assured for a *term of years,* except a Bonus should happen to be declared during the term for which the assurance is made; and then only in case the life should happen to become extinct *between the time of declaring such bonus and the end of the given term:* which, in most ordinary cases, can be but a *partial* advantage.

† The period of *ten* years (which has been adopted by the Society) is certainly too great an interval; since few persons can expect to enjoy the *full* value of their interest in the concern. A new plan, however, has just been adopted by the Society (which I shall presently mention) that will in some measure remove this objection, as far as the *old* members are concerned.

‡ See the method of making these valuations in Question XXXIV, page 457.

§ When it is considered that this *reserved third* is continually improved at interest, to the end of the next period, for the use of all the members, and then merges into the general stock to be again divided; it will appear that a person assuring at the Equitable Society, in its present established state (with its *reserved third* in abeyance), receives nearly the *whole* of his share of the *current* profits of the concern.

In the year 1786, the addition of 1½ per cent *on the sum assured*, which had been made to the claims in 1782, for *every annual payment* prior to that year, was increased to 2½ per cent. In the years 1791, 1793, and 1795, still further additions were made; amounting together to 4 per cent. In the year 1800, another addition, of 2 per cent, was made to the claims: and we have just seen a still further bonus declared, of 2½ per cent per annum, on all claims from assurances effected before December 7, 1809. So that at this time, the assurances of the year 1800 are increased by the *addition* of 25 per cent; those of 1790 are increased by 86 per cent; those of 1780 are increased by 180 per cent; those of 1770 are increased by 290 per cent; and those of 1762 are increased by 378 per cent. Or, in other words, (for I cannot be too explicit upon this important subject,) the executors of a person assured at this Office in the year 1800, for the sum of £1000 only, will on his decease be entitled to the sum of £1250: but, if the assurance has been effected in 1790, they will be entitled to £1860: if in 1780, to £2800: if in 1770, to £3900: and if in 1762 (when the Society was first established) they will be entitled to the enormous sum of £4780. Now, at *no other* Office (with the exception of the partial advantages derived from assuring at one or two of the Societies hereafter mentioned) would the executors be entitled to receive a shilling more than the sum of £1000 originally assured. Surely these *facts* speak sufficiently for themselves; and show, most decidedly, the impolicy of assuring at those Offices whose sole object is gain, and who consequently make no return of their immense premiums.

But the most important alteration in the plan of this Society was that which was brought forward and proposed at the General Court held December 7, 1809: where it was resolved that in future the addition of 2 per cent on the sum assured should be made to each existing policy *at the end of every year*, for the next 10 years: at the end of which period another valuation is to take place, when the surplus profits of the Society (after allowing for the bonus just alluded to) will be divided amongst the existing policies, in the usual manner. So numerous however were the applications for admission into the Society, on this proposal being made known to the public, that it was thought necessary and proper to amend it. At the next General Court (December 19, 1809) it was therefore resolved to limit this advantage to the persons *already* assured at the Office, and to such claims *only* as should arise prior to the next division of profits. Consequently, by this new plan, the executors of a person already assured at this Office will be entitled to receive (in addition to any other bonus which may have been declared) 2 per cent (on the original sum assured) for every year which has transpired since the date of the last division of profits and the time of his death. But, this benefit is not *yet* extended to members effecting assurances *subsequent* to the date above mentioned.

The proceedings of this Society have hitherto been founded on a firm and durable basis: and the Directors* have, in their management of it, shown the greatest prudence and discretion. Guiding themselves by the light of mathematical science (without which no establishment of this kind can be ever upheld) they have taken no step which can at all hazard its permanency or safety: ‘and, in order to secure the same prudence and discretion in the future conduct of its members, the Society has provided by certain laws, “That no allowance to claimants shall ever be made “without a previous investigation of its affairs, and “the concurrence of *four fifths* of its members at “three successive General Courts; and also that such “allowance shall in no instance exceed *two thirds* of “the clear surplus stock of the Society.” By these restrictions, as far as human precaution can operate, the danger of intemperate measures is avoided: and there is every reason to hope that an institution, founded on such liberal principles and defended by such wholesome provisions, will not only be secure, but continue to improve in credit and prosperity.†’

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* The Directors are chosen annually by the members of the Society: and four General Courts are held in the year; at which are exhibited accounts of the state of the Company, and at which are formed such rules and regulations as may be deemed necessary for its advancement. But, by a law made in January 1810, no person, admitted a member subsequent to December 19, 1809, has a voice at the general meetings unless he has been assured for 5 years in the sum of £2000 on the whole continuance of a life. Nor can any such person be chosen a director unless he has been assured for 5 years in the sum of £5000.

† See the history of this Society, brought down to the present time, in Price's *Obs. on Rev. Pay.* vol. 1, Chapter II, Section 7.

Other Societies.

§ 441. Soon after the establishment of the Equitable Assurance Company a number of societies sprang up, (all about the same time[*]), which assumed the false and alluring title of Institutions " for the *benefit* of Old Age." These institutions were, for the most part, gross impositions on the public; proceeding from ignorance or knavery, and encouraged by credulity and folly. The promoters of them were principally persons in the more advanced ages of life, and who imagined the schemes would last *their* time: they were consequently little anxious about the *durability* of the plan, and desirous only of getting a *numerous* body of subscribers, that their plunder might be the greater. Agreeably to this principle, the founders of these societies began so low as not to require so much as a *fourth* or a *fifth* part of the values of the annuities which they undertook to pay: and encouragement was also held out to *early* applicants. These base attempts to deceive and allure the ignorant and distressed (and which unfortunately succeeded too well) first induced

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[*] The Laudable Society, the Amicable Society of Annuitants, the Provident Society, the London Annuity Society, the Equitable Society of Annuitants, the Westminster Union Society, the London Union Society, the Consolidated Union Society, the Public Annuitant Society, the Rational Society, the Friendly Society of Annuitants, and many others (whose names are now lost or forgotten) were all established about the year 1770 and 1771.

Dr. Price to publish his *Observations on Reversionary Payments.* "On the first appearance however of this "work, the rage for establishing *new* Societies imme-"diately subsided; a *partial* reformation took place "in some of those which had been already formed: "and in a short time the greater part of them, con-"vinced of their mistakes, dissolved themselves. A "few, indeed, persevered in an obstinate adherence "to their original plans; but they have long since "exhibited a melancholy proof of their own folly, "and of the truth and justice of the admonitions which "had been wasted upon them.*"

The general practice of the promoters of those schemes was to attract the weak and credulous by holding out the most easy and advantageous terms. These arts succeeded wonderfully well at first; till the public became better informed on the subject; when they soon fell into disrepute and contempt. But although such specious proposals should always be viewed with suspicion, the public should be cautious not to fall into the opposite extreme, and suffer themselves to be enticed into the payment of exorbitant rates, under the delusive plea of stability and security. They ought to satisfy themselves that any extra charge which is made upon them (over and above the fair value, after allowing a proper and liberal compensation to the proprietors) is *sacredly* and *inviolably* appropriated to secure the permanency of the establishment, and to provide for those unfavourable events which sometimes arise and overturn the best contrived schemes.

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* *Introd.* to Dr. Price's *Obs. on Rev. Pay.*

Towards the end of the last, and the beginning of the present century, several new Public Companies were formed for the purpose of making Assurances on Lives, granting Annuities, &c: viz the *Westminster Society* in 1792; the *Pelican* in 1797; the *Globe* in 1799; the *Albion* in 1805; the *Rock* and the *Provident Institution* in 1806; and the *Eagle*, the *Hope*, the *London Life Association* and the *Atlas* in 1807. None of these societies however are confined *solely* to the granting of *Life* Assurances: since they all unite, thereto, the purchase or sale of *Annuities*; and five of them (in addition to the other two branches) are engaged in *Fire* Assurances. It is not my intention to enter into the nature and form of *all* these different Societies; since the major part of them do not (as far as the subject of this work is concerned) profess to hold out any peculiar advantages to the *public*.* But, as *two* of them in particular differ materially in this point from all the rest, I shall devote a few lines to their separate explanation and examination.

The Provident Institution.

§ 442. The *Provident Institution* was formed in 1806, with a view "to enable the industrious and economical

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* There is some trifling difference between them as to the *fees on admission*, and as to *military* persons: but, these are *paltry* considerations compared with those which I shall hereafter mention.

" to appropriate their savings most beneficially to their " different objects of prudence or affection; and to " encourage that laudable disposition by affording an " ample guarantee for the fulfilment of the benefits " proposed, *without curtailing them by commercial* " *views of profit.* With these objects *several hundred* " noblemen and gentlemen have entered into a sub- " scription of £250,000 to constitute an original ca- " pital *; and taking upon themselves the sole re- " sponsibility of the establishment, they render the in- " sured and annuitants *proprietors,* and entitle them " *equally* to participate in profits, after the expenses " of an economical arrangement are defrayed, in which " the Presidents, Directors, Trustees and Auditors " act gratuitously. Thus, by the specific sum insured, " and the division of contingent profits, every member " of this institution has the *fullest value possible* †, for " his payments, without being subject to any calls or " risk whatever."

This institution not only makes Assurances on lives, but also buys and sells Annuities, and grants Endowments for children. Their proposals, at first sight, appear to be the same as in the Equitable Society; since all the proprietors are stated to participate *equally* in the profits of the concern. There is however this material difference, that, in the Provident Institution, those profits are shared in conjunction with the *several hundred* noblemen and gentlemen who have *guarantied*[*] the capital; and who appear not to have wholly lost sight of the *commercial views of profit* in establishing this concern[†].

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* These subscribers are not *obliged* to insure *their own* lives, as is the case at *all the other* newly established Offices.

† This is a *false* assertion, as will evidently appear in the sequel.

The Rock.

§ 443. The *Rock Life Assurance Company* was formed also in the year 1806; and appears in some respects to have followed the plan of the Equitable Society: it is however in its nature essentially different. In the Rock, the assured are not *mutual* assurers one to the other: neither do they participate *equally* in the profits of the concern. For, though at certain periods (of not less than seven years[‡]) an estimate is made of those profits, and two thirds of the same, *after deducting the sum of five thousand pounds therefrom*[§], are divided amongst all the policies, as in the Equitable Society; yet the remaining third, instead of being improved for the benefit of *all* the parties, is appropriated to the use of the *proprietors only*. This third part is thus for ever *lost* to the rest of the assured; and, at the end of the next period, another third part is likewise taken away: which *continual subtraction* operates in the inverse method to that pursued by the Equitable Society, and must in a course of years make a very *material* difference in the amount to be added to the policies at each respective office.*

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[*] A small portion only of the capital has been actually *advanced*.

[†] See some observations on the *Rates* of this Society in § 374 and 446.

[‡] The *first* division of profits, however, is not to take place till after the expiration of *twelve* years from the establishment of the Society.

[§] This is provided for, in the Deed of Settlement, although it is not mentioned in the Prospectus of the Company. See a pamphlet issued by this Society under the erroneous and illusive title of *The Principles of Life Assurance explained.*

General Observations.

§ 444. With respect to the *remaining* institutions of this kind now in existence, I believe they are all formed with a view to *private* interest, and do not profess to hold out any peculiar advantage to the *public*.† It will be recollected that, a few years ago, there existed in this country a great rage for *Joint-Stock Companies;* and the public were much amused with the various proposals that daily issued from the press. The enormous profits that are made, by granting Life Assurances and Annuities, attracted the hopes and expectations of many persons; and several of the more recent Offices may ascribe their origin to these popular impressions. The success, attending the establishment of these

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* It will appear from this view of the subject that a person insuring at the Rock does not stand a chance of receiving even one *half* the additional profit that he would probably obtain by insuring at the Equitable. See the note in page 487.

† See the note in page 493.

Offices, and of similar undertakings, encouraged others to come forward, not only with the like proposals, but with various fanciful schemes and visionary projects: and there was every appearance that the times of the South-Sea Bubble were about to be revived. The same measures, however, that put a stop to the nefarious practices of that period, were now resorted to: the act of 6 Geo. I, c. 18,* after lying dormant above 80 years, was applied to the correction of this public grievance; and the decision of the Court of King's Bench, in the case of *The King* versus *Dodd*, put a stop to all these delusive projects. As the opinion of the Lord Chief Justice (Ellenborough) on that occasion cannot be made too public, and as it has some connection with the present subject, I shall make no apology for here inserting it.

" This case has been very fully argued, and the ap-" plication for an information has at least had this

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* By that statute it was declared that all the mischievous undertakings and attempts *therein described,* tending to the common grievance, prejudice and inconvenience of his Majesty's subjects, in their trade and commerce,—and all public subscriptions, receipts, payments, assignments or transfers, for the purpose of furthering, countenancing, or proceeding in such undertakings,—and more particularly the presuming to act as a corporate body, and pretending to raise and transfer any shares in such concerns without legal authority (either by Act of Parliament or Royal Charter)—should be illegal and void. All such undertakings are declared to be a public nuisance; and persons offending therein are not only subject to the usual fines in such cases, but are further liable to the penalties and forfeiture of the statute of *Præmunire*.

"good effect, that it has produced a full discussion "of the question, and has given a general notoriety "to the existence of the statute of the 6 Geo. I: so "that no person can hereafter pretend to say that it "is an obsolete law; and, on that account, no longer "to be enforced against such as offend against the "provisions of it. After a lapse, however, of eighty-"seven years since any authenticated proceeding has "been had upon this branch of the act, and when "other ways are still open to the party now applying "to put this act in force against offenders, the Court "(in the exercise of a sound discretion, under all the "circumstances of the case) will forbear to interfere "in this extraordinary manner. But, at the same "time we wish it to be understood that it is not be-"cause we think that the facts brought before us are "not within the penalty of the law: but we choose to "express ourselves with the greater reserve, because "the defendant may still be indicted, and the Court "may still be called (upon the removal of the indict-"ment by *certiorari*, or upon an information filed by "the Attorney-General) to give their opinion on this "very case. But (independent of the general tendency "of schemes, of the nature of the project now before "us, to occasion prejudice to the public) there is be-"sides in this prospectus a prominent feature of mis-"chief: for, it therein appears to be held out that no "person is to be accountable beyond the amount of "the share for which he shall subscribe, the conditions "of which are to be included in a Deed of Trust to

" be enrolled. But this is a *mischievous delusion,* " calculated to *ensnare* the unwary public. As to " the subscribers themselves, indeed, they may stipu- " late with *each other* for this contracted responsibility; " but as to the rest of the world, it is clear that *each* " partner is liable to the *whole amount of the debts* " *contracted by the partnership.* I forbear to com- " ment on lesser circumstances; such as the smallness " of the sum to be subscribed in the first instance,* " (which seems to carry an appearance of holding out " a *lure to the unwary*) and other features in the " case. But (considering that this is brought forward " after a lapse of so many years since any similar pro- " secution was instituted, and brought forward by a " party who does not profess to have been himself de- " luded by the project; and the statute having been " passed principally for the protection of unwary per- " sons from delusions of this kind) the Court think, " in the exercise of their discretion, that they should

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* Of all the Life Assurance Companies that have lately been established, the *Globe* is the only one that has paid up its subscription *in full.* The remaining ones have divided their capital into small shares, on which 10 per cent only has been paid. Thus, the *Eagle* professes to have a capital of TWO MILLIONS; whereas it in reality consists of no more than £200,000, which is divided into 40,000 shares of £5 each. And the *Rock,* which professes to have a capital of ONE MILLION, is divided into 50,000 shares of £2 each. A similar observation will apply to most of the other Societies. It is true that they have reserved a privilege of making a *call* for the remainder: but I hope no circumstances will ever arise, wherein their power or pretensions may be *disputed.*

" not now enforce the statute against this defendant, " at the relation of a person so circumstanced: leaving " the relator to the common law remedy by indict- " ment; or the defendant to be proceeded against " by His Majesty's Attorney-General *ex officio*, if he " should deem it adviseable for the protection of the " public. But, the Court think it is fit that this rule " should be discharged without costs. And they re- " commend it as a matter of prudence to the parties " concerned, that they should forbear to carry into " execution this mischievous project, or any other " speculative project of the like nature, *founded on* " *joint-stock and transferable shares*: and we hope " that this intimation will prevent others from engaging " in the like *mischievous and illegal projects*.*"

With respect to the constitutions of the several Assurance Companies above alluded to, they are all of them (with the exception of the *Amicable* and *Equitable* Societies) in one particular nearly the same. In each of them the Assured may be divided into two classes, Proprietors and Non-Proprietors. The former are those who hold shares in the concern and who divide all† the profit among themselves: and the latter consist of such other persons as have been induced from *ignorance*, *persuasion*, *or necessity* to effect assurances at those Offices; and who (although they pay the same premium on their policy as at the Equitable Society) will never receive a shilling more than the sum originally insured.

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* East's Reports, in Easter Term 1808, vol. ix. page 525.

† This must be taken with some limitation as far as it regards the *Provident Institution* and the *Rock*; agreeably to what I have above stated, when speaking of those companies. There is also the following advantage held out by the *Hope* Insurance Company: viz. " Persons, not being proprietors, who insure for £500 upon " their own lives, will, upon the payment of ten shillings per cent

For, it is necessary here to remark (and I think it an observation of considerable importance) that the *rates* of Assurances on lives do not differ, in the least, at any of the Offices: so that whether a person makes an assurance at the Equitable Society where the sum assured is continually *increasing* in value,* or whether he effects it at any other Office where no additional advantage is derived, *he pays precisely the same premium!* Surely this important fact cannot be sufficiently known by the public, else it is difficult to conceive how any of the newly established Offices should ever have been able to extend their business beyond the limits of their own proprietary.†

" (on their admission) beyond the premium paid for the Assurance, " be entitled, out of the profits of the Life Fund, to such interest " or dividend as may be paid on a £50 share to the several pro- " prietors thereof; and so on in like manner for every £500 so " insured. But in no case whatever are they to be responsible " for any claims which the Life Fund may be called upon to pay; " the Capital of the Company being of sufficient magnitude for " that purpose."

* See page 488.

† Every *proprietor*, in all these newly established Societies, is *obliged* to insure a given sum on his own, or on some other person's life.

Independent, however, of the superior *advantage* of insuring at the Equitable Society, there is a satisfaction arising from the consciousness of its superior *security*.* This security originated from, and is principally maintained by, the power which each member possesses (in his own right) of giving his vote and opinion on all the proceedings of the Society: so that no unfair advantage can be taken by any particular class of the assured. On the contrary, at all the other Offices† the management of the concern is in the hands of the Proprietors *only:* and the Non-Proprietors (or those who assure at the office without holding any share) have no voice whatever in the administration of the affairs of the Company. They may see the proprietors voting large dividends amongst themselves, and the capital of the society (which ought to be improved with the greatest economy) annually dwindling away: but, though this should be apparent to the most common observer, they would be, in a great measure, unable to relieve themselves. For, most persons, after continuing the payments on their policies through a series of years, would think it too serious an injury to withdraw; and would rather risk the issue of the event.

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* A person who makes an Assurance against *Fire* is content if the Office, at which he insures, possesses the *present* confidence of the public and of himself: since he may remove his policy to any other Office *whenever* he doubts its stability or responsibleness. Not so with a *Life* assurance: for, a policy of that kind, once entered into, cannot be removed without considerable loss and inconvenience to the party; and, as the *claim* may not become due for 20, 30 or 40 years (nay, perhaps for more than half a century) it therefore particularly behoves him to be satisfied not only with the *present* flourishing state of the society but with the prospect of its *permanency* and *future solvency*.

† Excepting the Amicable Society.

Now, though I by no means wish or intend to cast any reflection on the present managers of any of the more recent Societies, many of whom I *know* to be men of strict honour and integrity, and who (considering themselves equally trustees for the *public* as for the *Society*) would not give their sanction to any measure that was likely to endanger its security, or hazard its reputation; yet *their* voice may not always prevail against a numerous body of proprietors, neither can *their* services be perpetuated. And the sad experience of almost all preceding establishments of a similar kind, which have been formed with a view to the benefit of posterity, shows that this is not an *imaginary* evil only:[*] and that it becomes *every* person connected with such Societies to be particularly cautious that no attempts are made, either through ignorance or design, to risk the security of the establishment by grasping too eagerly at immediate or enormous profits. They should bear in mind the advice that was given to the Directors of the Equitable Society by a celebrated writer on these subjects, and the adoption of which has tended to raise that establishment to its present state of eminence. "They should consider what "distress would arise from the failure of the concern "at any future time; and what are the dangers which "ought to be carefully guarded against in order to "secure success."——"That it is not to be expected "that any society can meet with difficulties in its in"fancy; because, not till the end of many years, after "it has acquired its *maximum* of members, will the "*maximum* of yearly claimants and annuitants come "upon it. Should it, therefore, through inattention "to this remark, and the encouragement arising from "the possession of a large surplus, be led to check or "stop the increase of its stock [or to divide the pro"fits of the concern] *too soon,* the consequences "might be highly pernicious.*"——"That it is of "great importance to the safety of such a society that "its affairs should be under the inspection of able

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[*] Let any person read the nefarious practices alluded to by Dr. Price in his *Observations on Rev. Pay.* and by Mr. Dale in his *Calculations deduced from first Principles;* he will then be convinced that the members cannot keep too vigilant an eye over the proceedings of their society.

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* The distinguishing principle of the *London Life Association* is, that the profits resulting from its transactions shall be shared by the members *during life:* whereby the security of that establishment must be considerably impaired. For, "in matters of chance "it is impossible to say that an unfavourable run of events will "not come, which may hurt the best contrived scheme. The cal"culations only determine *probabilities*; and agreeably to these "it may be depended on that events will happen *on the whole.* But, "at *particular* periods, and in *particular* instances, great deviations "will often happen; and these deviations, at the commencement "of a scheme, must prove either very favourable or very unfa"vourable."

" mathematicians. Melancholy experience shows that " none but mathematicians are qualified for forming " and conducting schemes of this kind. In short; " dangerous mistakes may sometimes be committed, " if the affairs of such a society are not managed frugally, carefully, and prudently.*"

Rates for Life Assurances.

§ 445. I cannot conclude this chapter without some observations on the *Rates* of Life Assurances published by the several Companies above alluded to: and, as those rates are the *same* in all, my remarks will *equally* apply to each of those different societies; excepting perhaps the *Equitable*, the nature of whose establishment renders it not of so much moment what the rates are, provided every member pays in proportion.* The promoters of that concern showed their sound judgement and good sense in adopting such tables of the probabilities of life as were rather *below* than *above* the truth. They first made use of Mr. Simpson's Table for *London*; but, finding these too inaccurate, they changed them for the *Northampton* Table. Experience has confirmed the propriety of their choice; but, at the same time, has proved to us that the *Northampton* Table is by no means a *corret* index of the rate of human mortality amongst the members of such a society.† It has, however, been adopted by all the *other* offices, under the delusive plea that "experience "has shown it to be the most just and liberal scale, "whereby to estimate a fair value between the two "parties." But, however applicable the *Northampton* Table may be to the use of a society *constituted as the*

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* Price's *Obs. on Rev. Pay.* vol. 1, page 179. The same author (in pointing out, in another place, the dangers to which a Society of this kind is exposed) remarks, " that there are no questions, the " solution of which requires a stricter attention, or greater skill in " investigation, than some in the doctrine of Assurances. Difficult " questions are sometimes brought to the Society; and the Di" rectors, not being themselves mathematicians, are under a ne" cessity, in making their demands, of being governed by their " Actuary: and should he happen to be unqualified, he must make " mistakes, and either the Public or the Society will be injured. " In short, the Society can scarcely be sensible enough of the " importance of both abilities and probity in the servants it em" ploys: nor, therefore, of the particular reason there is for guarding " their places against the applications of candidates who, on any " future vacancies, may endeavour to intrude themselves by their " connexions or influence."

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[*] I should likewise except the *Amicable Society,* whose plan and constitution differ so materially from the others.

[†] Mr. Morgan says (Price's *Obs. on Rev. Pay.* vol. 1, page 183) that the decrements of life in the Equitable Society, from the year 1768 to 1800, compared with the decrements of life in the *Northampton* Table, appear to have been from the age of

10 to 20 in the ratio of	1 to 2
20 .. 30	1 to 2
30 .. 40	3 to 5
40 .. 50	3 to 5
50 .. 60	5 to 7
60 .. 80	4 to 5
All ages together	2 to 3

Equitable is, it certainly is not entitled to the same degree of approbation as affording a fair estimate of the value of annuities and assurances between the public and a *trading* company; nor even as between two individuals, unless under particular circumstances. A more just and equitable scale *ought* to be adopted by those societies who do not make any return to the assured, of the vast profits that arise from this species of daily traffic; and would tend more to the increase of their business in this way, and would likewise be more honoûrable to themselves, than the disgraceful practice of *bribing* solicitors, agents and others to effect assurances at their Offices: thereby notoriously inducing those parties to *sacrifice* the interest of their employers and their friends.* For, the money which is applied to this base purpose can be considered in no other light than as *unjustly* taken from the pockets of the *assured*; and would be more properly and more equitably employed in being appropriated towards a reduction of the rate of assurance: since, if the Company can afford to allow it to the *agent*, it surely can afford to allow it to the principal; and it evidently belongs more *justly* to the latter than to the former.

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* Many of the public Companies, who do not make any return of the profits to the assured, allow a *liberal premium* (generally 5 per cent on the payment made) to any person who will procure an insurance to be effected at their Office: and this commission is *also* allowed to *any* person who makes the *annual* payment *provided it be not the party himself!!!* An artifice which is easily seen through: but which opens such a door to fraud and imposition, that it cannot be too severely reprobated. And however much it may be sanctioned by the Directors in their *public* capacity, we are all aware what their emotions would be if they discovered any of their tradesmen tampering with *their own* servants in this opprobrious manner: since they must well know *who* would eventually pay for it. I omit to give the names of those companies who have adopted this nefarious practice, under the hope that such a mean and improper artifice will not be encouraged in future.

The Tables of the *Rates* of Life Assurances, as published by the several Companies, are precisely the *same* in each; and are usually *three* in number: viz. one for *Single* lives, one for *Joint* lives, and the other on the *Contingency* that one life shall die before another life.* These rates are all deduced from the *Northampton* observations, and at the rate of *three* per cent interest. By thus computing the values from the *lowest* probabilities of life, and at the *lowest* rate of interest, the Rates become, in most cases, full a *third* more than they ought to be when calculated from the more correct tables of the probabilities of life as observed in *Sweden*, at the rate of *four* per cent interest; and still more do they differ from the true value as deduced from the probabilities of life as observed by *M. De Parcieux*, at the same rate of interest.

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* These Tables are LI, LII, and LIII at the end; and which may be had at any of the Assurance Offices. On a comparison of them, as issued by the different Offices, they will be found to be all alike: there being only a trifling difference of a *penny* or *two-pence* per cent in any of them; which is sometimes in *excess* and sometimes in *defect*.

Now, since the rate of human mortality, amongst the members assured at such Societies, is by no means correctly indicated by the *Northampton** table of observations, but approximates more nearly to the observations made in *Sweden*, or to those made by *M. De Parcieux*, the public may form a tolerably correct idea of the immense† profit that is made by this species of daily traffic; and may learn, to their own cost, the impolicy of insuring at those Offices whose sole object is gain, and who consequently make no return of any part of these exorbitant demands. A single instance will illustrate this.

A person aged 20 is desirous of assuring his own life for £5000: the sum demanded for this purpose by *all* the Assurance Companies is £2140: 2: 0. But the true value of such sum, as deduced from the *Sweden* Observations, is no more than £1422 : 10 : 0, if we take the rate of interest at 4 per cent; and no more than £1128: 10: 0, if we take the rate of interest at 5 per cent: or when deduced from the observations of *M. De Parcieux*, it is no more than £1358 : 2 : 0, taking interest at 4 per cent; and no more than £1078: 16: 0, taking interest at 5 per cent. A person, therefore, of this age who insures the above sum at any of those Offices, which make no return of any part of the premium, may be considered as *throwing away* between seven hundred and a thousand pounds [*]. I say that this sum is thrown away by insuring at *such* Offices, because, by effecting the same assurance at the *Equitable Society*, the money thus paid down is carried to a common stock, and the *whole* [†] of the profits of the concern (after the expenses of an economical establishment are defrayed) are from time to time divided amongst *all* the persons assured, and added to their policies: so that the executors of a person assured at *that* office have a chance of receiving (at his death) *considerably more* [‡] than the sum originally insured. The public, I presume, are not aware of this important fact; otherwise it is difficult to conceive how the other

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* It is in the ratio of only 2 to 3. See the note in page 506.

† Some idea may be formed of these enormous gains, from the large sums which have occasionally been added to the policies at the Equitable Society; and which have amounted to above *one million and a half* in present money. See page 468.

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[*] These calculations are deduced from the supposition that the *whole* premium is paid down immediately in *one payment:* but, nearly the same differences will arise if we suppose the premiums to be made *annually*. For the annual payment which would be required, by all the Offices, for the assurance above alluded to, is £108 : 19 : 0: whereas the value of the same, deduced from the observations of *M. De Parcieux*, is £71 : 14 : 3, if interest be taken at 4 per cent, and only £65 : 10 : 0, if interest be taken at 5 per cent.

[†] Every person now insuring at that Office may be considered as partaking of the *whole* of the profits of the concern; for the reasons given in the note in page 487.

[‡] They have a chance of receiving considerably more, for two reasons: in the first place the Society can generally make more than *three* per cent interest of their money; and secondly, the lives do not die off so fast as the computations suppose. So that the money is not only improved at a better rate of interest, but also for a longer period.

Offices should ever have been able to effect a single policy, except such as the Equitable Society had refused.

The same observations will apply to the case of *Joint* lives; since the values are here also deduced from the *Northampton* table of observations, and at the rate of *three* per cent interest. But, if we take the case of *Contingent* assurances (that is, of an assurance made on a given life, or the contingency that it dies before another) we shall find still greater cause for censure. For, independent of the rates being computed from the *lowest* probabilites of life and at the *lowest* rate of interest, they are also deduced from a rule given by Mr. Simpson *, which produces a result that is oftentimes more than *one third* of the true value too much, even when computed from the *Northampton* table, at 3 per cent interest!—A single instance will confirm this also.

A person, 10 years of age, is desirous of assuring the sum of £100 on his life, on the contingency that he dies before another person aged 60. The sum which would be demanded by *all* the Assurance Offices (not even excepting the *Equitable*) is £12 : 18 : 6 in a single payment, or £1 : 6 : 0 in annual payments during the joint lives. But the true value which *ought* to be given for the same (even on the supposition that the rate of interest is *no more* than 3 per cent, and the probabilities of living the *same* as observed at *North-*

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* See the Scholium in page 446.

ampton) is only £10 : 13 : 0* in a *single* payment, and £1 : 1 : 5 in *annual* payments: consequently the Offices demand about a *fourth* part of the true value *more* than (*on their own data*) is just and equitable. If the value, however, had been deduced from the probabilities of living as observed by *M. De Parcieux*, and at the rate of 4½ per cent interest, it would be £9 : 3 : 1 in a single payment, and only *eighteen shillings and ninepence* in annual payments: which makes the Office rate, in this case, nearly one *half* the true value too much!

Mr. Morgan has taken considerable pains to prove that Mr. Simpson's rule for finding the value of these contingent reversions is *exceedingly defective* †; and that it oftentimes leads to conclusions too erroneous to be overlooked. Now, since the *true* values can in all cases be obtained with so little trouble, it is somewhat singular that the incorrect values, in Table LIII above alluded to, should still be adopted (not only by the *Equitable Society*, but also by every other Office in London) for the purpose of determining the sums that are required for effecting assurances on the contingency here mentioned. Amongst the numerous societies that have lately been established, is there no one Actuary that has the confidence to propose a *new* table of the value of such assurances, *founded on a true and proper basis*: or *will* the several Companies still persevere in their unjust and illiberal demands? Surely their profits must be sufficiently great by taking the *lowest* rate of interest, and the *lowest* probabilities of living as the basis of their calculations; without adding thereto the *unfair* advantage arising from the use of *inaccurate rules* *.

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* I take this sum from Mr. Morgan's calculation in *Phil. Trans.* for 1788, page 344; who says that he has compared it with the sum deduced from the *exact* values of annuities on the joint lives involved in the solution.

† See *Phil. Trans.* ibid. The *singular* formula however, from which Mr. Morgan has deduced his values, has at length been amended by him in the *last* edition of Dr. Price's *Obs. on Rev. Pay.* Note (O). See what has been said in the note in page 187 of the present work.

The following tables (given by Mr. Morgan himself †) of the present value of £100 payable on the decease of A, provided B be then alive, will show "how far Mr. Simpson's approximation, the only rule "now in use [*by the Assurance Offices*], may be de-"pended upon."

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* These observations will, in a great measure, apply also to the Equitable Society; since those, who assure on this contingency, pay more *in proportion* than the rest of the members: and it is curious to observe how servilely the *other* Offices have followed their example. See what has been already said upon this subject in the note in page 188.

† See *Phil. Trans.* ibid.

Interest 3 per cent, *Northampton* Observations.

Age of A	Age of B	Value by Simpson's rule.	Value by correct rule.	Age of A	Age of B	Value by Simpson's rule.	Value by correct rule.
10	10	24·75	24·75	20	20	27·96	27·96
10	20	23·50	22·11	20	50	19·29	18·65
10	30	21·47	19·84	30	30	30·21	30·21
10	40	19·07	17·10	30	60	18·19	17·51
10	50	16·21	14·04	40	40	32·87	32·87
10	60	12·93	10·65	40	60	22·78	21·92
10	70	9·15	7·07	40	70	15·78	15·35

Interest 4 per cent, *Sweden* Observations.

Age of A	Age of B	Value by Simpson's rule.	Value by correct rule.	Age of A	Age of B	Value by Simpson's rule.	Value by correct rule.
14	20	17·82	15·42	40	40	26·99	26·99
16	40	16·23	13·71	40	76	9·81	9·21
20	20	19·84	19·84	42	60	19·58	16·11
24	60	13·01	9·39	52	76	14·00	12·58
28	40	20·44	17·60	60	60	36·34	36·34
36	60	16·81	12·29	64	76	22·81	23·81

It will be seen by this comparison that, when the ages of the two lives are *equal*, the exact values are found by either rule *: but when there is any considerable inequality between the ages of the two lives, the values by Mr. Simpson's rule are in many cases

* Owing to some strange error in his calculation, Mr. Morgan has made the values deduced from Mr. Simpson's formula (according to the Sweden observations and when the lives are equal) *different* from those obtained by the correct rule: whereas it must be evident, from an inspection of the formula in the note in page 446, that they ought in such case to be *precisely the same.* The correction of this error is the only alteration I have made in the Table.

so wide from the truth, as to show the impropriety of their being used by any office, *professing* to be guided by mathematical principles.

Rates of Life Annuities.

§ 446. The observations which I have made in page 361 will apply with equal force to the terms on which the *Provident Institution* grants *deferred* Life annuities. For instance: the sum of £116 : 8 : 0 is demanded for an annuity of £20 on the life of a person, now aged 20, to commence when he is 40 years of age. Now the value of the same annuity, on the life of a person aged 40, is (at the same Office) estimated at £296. But £116 : 8 : 0 put out to interest at 5 per cent for 20 years would amount to £308 : 16 : 10; or to near £13 *more* than the sum for which he might *then* obtain the same annuity, without the risk of losing his money *in the mean time !!!*

With respect to the value of Life Annuities in general, at this Office, (and which is the only one, except the Royal Exchange, that has hitherto published its rates), I would observe that, like the values adopted by that Corporation, they are too *low* in the younger, and too *high* in the older periods of life. The cause of which anomaly is too apparent: since life annuities are generally purchased by those of more *advanced ages*, and are seldom resorted to by the *young*. I would however remark that those persons who are desirous of *purchasing* annuities on their own lives, or on any other person's life, or on two lives with benefit of survivorship, have now * an opportunity of doing it with much greater security and advantage, than at *any* of the Offices above alluded to. For, by the act 48 Geo. III, c. 142, any person may exchange *Reduced* or *Consolidated* 3 per cent stock for an annuity on the life of a nominee not younger than 35 and not older than 75 years of age. And as the terms, on which this exchange can be made, are always more advantageous to the party than by a direct purchase of such annuities at any of the Offices; it is probable that this plan will, for the future, be the only one resorted to, in such cases, as long as it remains in effect †. This observation, however, applies only to such persons as wish to *purchase* annuities. But there are many others who are obliged, from various causes, to *sell* or grant annuities on their own, or on other persons', lives: and several of the Offices are in the habit of purchasing the same; or, of advancing money on such contingencies. I am unacquainted with the *terms* on which this part of their business is conducted: neither do I suppose that they are nicely weighed by that class of persons who are driven to such an unfortunate and ruinous measure,

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* In my *Doctrine of Interest and Annuities,* page 106, I suggested the propriety and advantage to Government in exchanging some of the Public Stocks for *Life* Annuities: and I proposed to enter more at large on this subject at some future opportunity. But, as that plan has since been adopted on so fair and liberal a scale, I can only add my best wishes for its success.

† The *rate* at which the exchange can be made depends on the price of the stock on the day *preceding* the day of transfer: and the exchange may be effected at any time when the price of the 3 per cents is between 60 and 81; the terms varying according to every *one per cent* difference in the market price of the stock. The limits of this work will not enable me to insert the whole of the Table given in the act: I shall therefore subjoin only a few of the values of annuities on Single lives according to *three* different prices of stock; to which I shall add the terms of the Royal Exchange Assurance Annuity Company and of the Provident Institution (the only ones that have hitherto published their rates), whereby an estimate may be formed of their comparative advantages. I would previously, however, observe that the annuities granted by Government and by the Provident Institution are pay-

Conclusion.

§ 447. Before I conclude this subject I ought to mention that, in consequence of the pernicious practice of raising money by the sale of Life Annuities,

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able *half-yearly:* whereas those granted by the Royal Exchange are payable *quarterly*. But this does not make any *material* difference in the true value.

Age.	Royal Exchange Rates.	Provident Institution Rates.	Government Rates.		
			4 per cent. or Stock at 75.	4½ per cent. or Stock 66⅔.	5 per cent. or Stock at 60.
35	15·380	15·600	14·563	13·878	13·043
40	14·700	14·800	13·761	13·072	12·371
45	13·700	13·700	12·821	12·232	11·650
50	12·820	12·600	11·719	11·299	10·714
55	12·050	11·495	10·638	10·256	9·836
60	10·870	10·250	9·494	9·009	8·889
65	9·800	9·000	8·242	8·081	7·792
70	8·770	7·750	6·849	6·734	6·522
75	7·750	6·500	5·395	5·333	5·195

which was much promoted by the secrecy with which such transactions were conducted, it was enacted by 17 Geo. III, c. 26 (commonly called the *Annuity Act*) that a memorial of all deeds, bonds, &c. for granting Life Annuities shall, within 20 days of the execution thereof, be enrolled in the Court of Chancery; which memorial shall not only contain the date, names of *all the parties* (and for whom any of them are trustees), and of all the witnesses, but shall likewise set forth the amount of the annuity, the consideration (which shall be in money only *), and the name of the annuitant: otherwise such deed, &c. shall be null and void †.

This being the case, it would be almost impossible for any of those societies, of the nature of Joint-Stock Companies, to deal in this kind of securities; since the names of *all the proprietors* (amounting to some hundreds, or perhaps to some thousands‡) must be inserted in the deed: and it would in fact be difficult, if not impossible, to collect all those names, at the time of making any one contract, owing to the fluctuating state of the proprietary. To obviate this inconvenience, some of those Societies have obtained Acts of Parliament with a view to enable them to enrol such memorials in the names of the parties to such deeds, &c; and with an intention that such enrolment might (all other requisites of the Annuity Act being duly complied with) be as good and effectual, to all intents and purposes, as if the names of *all* the parties interested were inserted therein.

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* Bank notes, a cheque on a banker, a promissory note, or a bill of exchange are held to be a good consideration provided they are *paid when due*, and also provided that they are *fully set forth* in the deeds.

† There are some exceptions to this act; amongst which are annuities granted *by* any Corporate Body, or under any authority or trust created by Act of Parliament.

‡ See the note in page 499.

The Societies, that have obtained these illusive* Acts, are the *Globe*, the *Albion*, the *London Life Association*, the *Pelican*, and the *Provident Institution*: and there is a clause in those acts whereby these Companies are enabled to *prosecute* and to *sue* in the name of one of their officers; but they have not the privilege of *being sued* in the same manner. And it is

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[*] See the Acts 47 Geo. III, Sess. 1, c. xxx, xxxi, xxxii, xxxiii and xxxiv. Owing to a strange oversight, these Acts became of no effect, almost as soon as formed; and, in order to render them of any real service to the parties, it was found necessary to have them amended. The *Globe*, the *Albion* and the *Pelican* therefore obtained new Acts for that purpose, in the very same year; being 47 Geo. III, Sess. 2, c. lxxxvi, lxxxvii and lxxxviii: but the *London Life Association* and the *Provident Institution* remain in their original doubtful state. These new Acts, however, being still open to some objections, were further amended and *explained* (at least, the titles *so* express it) by 49 Geo. III, c. cxxiii, cxxiv and cxxv.

Under shelter of these statutes, the *Albion* boldly claims the ambiguous quality of being "Empowered by Act of Parliament:" but the *Provident Institution*, with *less* right and more effrontery, falsely asserts that it is "specially empowered by Act of Parliament "for the Insurance of Lives, and the grant and purchase of Annui-"ties." Surely this artifice is unworthy the respectable names at the head of those Societies.

moreover expressly provided by each Act "that nothing "therein contained shall extend, or be deemed, construed, or taken to extend to *incorporate* the said "society or partnership; or to relieve or discharge the "said society or partnership (or any of the members "thereof, or subscribers thereto respectively) from any "responsibility, contracts, duties or obligations whatsoever; which, by law, they may (now, or at any "time hereafter) be subject or liable to, either as between such society or partnership and others; or "between the (or any of the) individual members of "such society or partnership and others; or among "themselves; or in any other manner whatsoever."

I cannot close this chapter without observing that, by statute 14 Geo. III, c. 48, it is enacted that no Assurance shall be made on lives, wherein the party assured hath no interest; that in all policies the name of such interested party shall be inserted; and nothing more shall be recovered thereon than the amount of the interest of the assured. And it has lately been decided that, although an interest existed *up to the very time that the claim became due*, yet, if such interest be *afterwards* destroyed, the money could not be recovered. See (in East's Reports, Vol. 9, page 72) the case of *Godsall and others* versus *Boldero and others*, three of the directors of the *Pelican* Life Assurance Company, who refused to pay the sum of £500, insured on the life of the late Mr. Pitt, under the plea that his debts had been paid by Parliament:

TABLES.

TABLE I.

Showing the Number of Persons *living*, at the

Ages	Lond.	Stockholm		Vienna	Berlin	Norwich	Northampt.	Chester	
		Males	Fem.					Males	Fem.
0	1000	1000	1000	1000	1000	1000	1000	1000	1000
1	680	577	611	542	633	730	743	773	828
2	548	497	521	471	528	595	625	678	739
3	492	443	468	430	485	544	582	624	683
4	452	405	433	400	434	517	553	588	648
5	426	381	413	377	403	498	536	571	623
6	410	366	398	357	387	481	521	556	609
7	397	355	386	344	376	467	509	543	601
8	388	346	377	337	367	457	499	537	596
9	380	338	370	331	361	447	492	528	591
10	373	330	364	327	356	440	487	524	588
11	367	325	359	322	353	434	483	521	585
12	361	321	356	318	350	429	478	519	583
13	356	317	354	314	347	424	474	516	579
14	351	314	351	310	344	419	470	513	576
15	347	310	348	306	341	415	465	511	572
16	343	306	345	302	338	411	461	506	568
17	338	301	342	299	335	407	457	501	563
18	334	296	339	295	332	403	452	496	558
19	329	291	335	291	328	398	446	490	553
20	325	285	331	288	324	394	441	485	547
21	321	279	327	284	320	389	434	479	543
22	316	273	323	280	315	384	428	473	538
23	310	267	319	276	310	379	421	467	333
24	305	261	315	273	305	374	415	461	528
25	299	254	311	269	297	369	409	454	523
26	294	247	306	265	293	364	402	448	515
27	288	240	302	261	287	358	396	441	508
28	283	233	297	256	281	352	389	434	500
29	278	226	291	251	275	346	383	428	493
30	272	219	285	247	269	341	376	422	485
31	266	212	279	243	264	336	370	417	479
32	260	205	273	239	259	331	364	412	473
33	254	198	267	235	254	326	357	407	467
34	248	192	261	231	249	321	351	402	461
35	242	185	254	226	243	316	344	397	455

TABLE I.

several Ages, and Places therein mentioned.

Ages	Bres-law	Sweden			Parish in Brand.	Holy Cross	Vauc	Annts in Holland	[illegible] in France
		Males	Fem.	Both					
0	1000	1000	1000	1000	1000	1000	1000	1000	1000
1	769	770	791	780	775	817	811	804	745
2	658	720	739	730	718	754	765	768	709
3	614	686	704	695	687	708	735	736	682
4	585	662	679	671	664	677	715	[illegible]	662
5	563	647	666	656	642	651	701	689	647
6	546	635	654	644	622	630	688	676	634
7	532	624	643	634	607	614	677	664	624
8	523	615	635	625	595	601	667	652	615
9	515	608	628	618	585	594	659	646	607
10	508	601	622	611	577	589	653	639	600
11	502	596	616	606	570	585	648	633	595
12	497	591	612	602	564	582	643	627	590
13	492	587	608	597	559	579	639	621	585
14	488	583	604	594	554	576	635	616	581
15	483	579	601	590	549	573	631	611	578
16	479	575	597	586	544	569	626	606	574
17	474	571	593	582	539	565	622	601	570
18	470	567	589	578	535	561	618	596	565
19	465	563	585	574	531	557	614	590	561
20	461	558	581	570	527	553	610	584	556
21	456	553	577	565	522	548	606	577	551
22	451	548	572	560	517	542	602	571	545
23	446	543	568	555	512	537	597	566	540
24	441	538	564	551	507	531	592	559	534
25	436	532	559	546	502	525	587	551	529
26	431	527	555	541	498	519	582	543	523
27	426	521	550	535	495	512	577	535	517
28	421	516	544	530	492	506	572	526	512
29	415	510	539	525	489	500	567	517	506
30	409	505	533	519	486	494	563	508	500
31	403	499	527	513	482	489	558	499	495
32	397	493	521	507	477	483	553	490	490
33	391	487	515	501	472	478	548	482	484
34	384	481	508	495	467	472	544	474	479
35	377	475	502	488	462	466	539	467	474

TABLE I Continued.

Ages	Lond.	Stockholm		Vienna	Berlin	Norwich	Northampt.	Chester	
		Males	Fem.					Males	Fem.
36	236	179	248	221	237	311	338	391	448
37	230	172	242	216	230	306	331	385	442
38	224	166	236	211	223	300	325	379	435
39	218	160	230	205	216	295	318	373	429
40	212	154	223	199	209	290	312	366	422
41	207	148	217	194	203	284	305	360	415
42	201	142	210	189	197	279	299	352	408
43	194	136	204	185	192	274	292	345	401
44	187	130	197	181	187	268	285	337	394
45	180	124	190	176	182	263	279	329	387
46	174	119	184	171	177	257	272	322	380
47	167	113	177	165	172	251	265	314	373
48	159	108	171	159	167	245	259	306	366
49	153	102	164	153	162	239	252	298	359
50	147	97	158	147	157	233	245	290	352
51	141	91	152	142	152	227	238	281	345
52	135	86	146	137	147	221	231	273	338
53	130	81	141	133	142	215	224	264	331
54	125	76	135	128	137	208	217	256	325
55	120	71	130	123	132	202	210	249	318
56	116	67	125	117	127	195	203	241	312
57	111	62	120	111	121	188	196	234	306
58	106	58	115	106	115	181	189	226	300
59	101	54	110	101	109	175	182	219	293
60	96	50	105	96	103	168	175	211	286
61	92	46	100	91	97	161	168	201	277
62	87	42	94	87	92	154	161	190	265
63	83	38	88	82	88	147	154	178	253
64	78	35	82	77	84	139	147	167	242
65	74	31	77	72	80	132	140	156	232
66	70	28	72	67	75	124	133	148	224
67	65	26	67	62	70	116	126	141	217
68	61	23	62	57	65	109	119	136	210
69	56	21	57	52	60	101	113	130	202
70	52	20	52	48	55	94	106	123	193
71	47	17	47	44	51	86	99	115	181
72	43	15	42	40	47	79	92	103	167
73	39	13	38	36	43	73	85	92	153
74	35	11	33	33	39	66	78	81	139
75	32	10	28	30	35	59	71	72	127

TABLE I Continued.

Ages	Bres-law	Sweden			Parish in Brand.	Holy Cross	Vaud	Aunts. in Holland	Tonts in France
		Males	Fem.	Both					
36	370	469	496	482	456	460	533	460	460
37	363	463	490	477	450	453	527	453	464
38	356	457	485	471	444	447	520	446	459
39	349	451	479	465	438	441	513	439	454
40	342	445	473	459	432	435	506	432	449
41	335	438	467	453	427	428	500	425	444
42	328	431	459	445	422	422	494	419	439
43	321	423	452	437	417	416	488	413	434
44	314	415	444	430	412	409	482	407	429
45	307	407	437	422	407	402	476	400	424
46	299	399	429	414	400	394	469	393	419
47	291	391	423	407	394	387	461	386	413
48	283	383	416	400	388	380	451	378	408
49	275	375	410	392	381	373	441	370	402
50	267	367	403	385	374	365	431	362	396
51	259	357	395	376	367	358	422	354	390
52	250	348	387	367	359	351	414	345	384
53	241	338	379	358	351	344	406	336	378
54	232	329	370	349	343	335	397	327	371
55	224	319	362	340	334	327	388	318	363
56	216	310	353	331	324	319	377	309	355
57	209	300	345	322	314	310	364	300	346
58	201	290	336	312	304	301	348	291	338
59	193	280	327	303	293	292	331	282	329
60	186	270	317	293	282	283	314	273	319
61	178	260	306	282	271	273	299	264	309
62	170	249	294	271	260	264	286	255	299
63	163	237	282	259	248	255	274	245	288
64	155	225	270	247	236	245	262	235	278
65	147	214	258	235	224	236	250	225	267
66	140	203	246	224	213	226	236	215	256
67	132	191	234	212	202	216	220	205	245
68	124	179	222	200	190	206	202	195	234
69	117	167	210	187	178	195	184	185	222
70	109	154	198	175	166	185	168	175	211
71	101	142	185	162	153	175	153	165	199
72	93	129	171	149	138	164	140	155	187
73	85	117	156	135	122	155	129	145	175
74	77	105	140	121	107	144	119	135	162
75	69	94	125	108	93	134	109	125	148

TABLE I Continued.

Ages	Lond.	Stockholm		Vienna	Berlin	Nor-wich	North-ampt.	Chester	
		Males	Fem.					Males	Fem.
76	28	8	23	27	32	52	65	64	116
77	25	6	19	24	29	46	58	58	106
78	22	5	16	21	26	40	52	52	96
79	19	4	13	18	23	35	46	47	86
80	17	3	10	16	20	31	40	41	76
81	14	2	8	14	18	27	35	36	66
82	12	2	6	12	16	24	30	31	57
83	10	1	4	10	14	20	25	26	47
84	8	1	3	8	12	17	20	22	37
85	7	1	2	7	10	14	16	19	29
86	6	0	1	6	8	12	12	15	23
87	5	0	1	5	7	9	9	13	19
88	4	0	0	4	6	8	7	11	16
89	3	0	0	3	5	6	5	9	14
90	2	0	0	2	4	4	4	7	13
91	1	0	0	1	3	2	3	6	11
92	0	0	0	0	2	1	2	4	9
93	0	0	0	0	1	0	1	2	7
94	0	0	0	0	0	0	1	1	5
95	0	0	0	0	0	0	0	0	3
96	0	0	0	0	0	0	0	0	2
97	0	0	0	0	0	0	0	0	1

TABLE I Continued.

Ages	Bres law	Sweden			Parish in Brand	Holy Cross	Vaud	Annts. in Holland	Tonts. in France
		Males	Fem.	Both					
76	61	84	111	96	80	124	98	114	134
77	53	74	98	84	68	115	85	103	120
78	45	65	86	75	59	106	71	92	106
79	38	56	75	65	51	96	58	82	94
80	32	48	65	56	44	87	46	72	81
81	26	41	55	47	38	78	36	62	70
82	22	34	46	38	32	69	29	53	59
83	18	27	38	31	25	61	24	45	49
84	15	21	30	24	21	53	20	38	40
85	12	16	22	19	15	44	17	31	33
86	9	12	17	14	11	36	14	25	26
87	6	9	13	11	8	29	11	19	21
88	4	7	10	8	6	23	9	14	16
89	2	5	8	6	4	18	7	10	12
90	1	4	6	5	3	13	5	7	8
91	0	3	4	3	2	10	4	5	5
92	0	2	3	2	1	8	3	4	3
93	0	1	2	1	0	6	2	2	1
94	0	0	1	0	0	4	1	1	1
95	0	0	1	0	0	2	0	0	0
96	0	0	0	0	0	1	0	0	0
97	0	0	0	0	0	0	0	0	0

TABLE II.

Showing the *Expectation of Life*, at the

Ages	Lond.	Stockholm		Vienna	Berlin	Nor-wich	North-ampt.	Chester	
		Males	Fem.					Males	Fem.
Birth.	17·90	14·25	18·10	16·37	17·85	23·16	25·18	28·13	33·27
5	35·28	31·05	37·12	36·54	28·67	40·22	40·84	43·20	47·44
10	34·91	30·00	36·89	37·02	37·15	40·26	39·78	41·92	45·17
15	32·32	26·74	33·43	34·11	33·65	37·50	36·51	38·05	41·36
20	29 37	23·85	30·01	31·39	30·34	34·37	33·43	34·86	38·10
25	26·66	21·40	26·80	28·32	27·47	31·56	30·85	32·00	34·78
30	24·11	19·42	23·98	25·62	25·25	28·93	28·27	29·25	32·27
35	21·76	17·58	21·62	22·66	22·76	26·06	25·68	25·97	29·26
40	19·50	15·61	19·25	20·49	20·91	23·18	23·08	22·92	26·37
45	17 63	13·78	17·17	17·82	18·85	20·30	20·52	20·20	23·50
50	15·84	11·95	15·12	15·88	16·40	17·55	17·99	17·64	20·62
55	13·91	10·30	12·89	13·50	14·14	14·88	15·58	15·14	17·52
60	11·69	8·69	10·45	11·65	12·49	12·36	13·21	12·36	14·20
65	9·69	7·39	8·39	9·51	10·48	10·06	10·88	10·79	11·94
70	8·00	5·81	6·16	8 30	8·69	8·12	8·60	8·05	8·81
75	6·27	4·09	4·39	6·37	7·08	6·44	6·54	7·00	7·14
80	4·86	3·20	3·09	5·50	6·07	5·15	4·75	5·43	5·20
85	3·04	1·78	2·02	3·33	4·50	3·50	3·37	4·25	4·81
90	·00	·00	·00	1·50	2·83	1·50	2·41	2·50	3·46
95	·00	·00	·00	·00	·00	·00	·75	·00	1·21

TABLE II.

several Ages, and Places therein mentioned.

Ages	Breslaw	Sweden			Parish in Brand.	Holy Cross	Vaud	Annts in Holland	Tonts. in France
		Males	Fem.	Both					
Birth.	26·60	33·20	35·70	34·42	32·51	33·93	37·56	34·47	34·79
5	41·47	45·62	48·00	46·73	45·08	46·30	47·11	44·45	48·19
10	40·49	43·04	46·25	45·07	44·93	46·00	45·66	42·71	46·76
15	37·40	40·56	42·76	41·64	42·16	42·25	42·17	39·55	43·46
20	34·15	36·95	39·15	38·02	38·70	38·66	38·53	36·31	40·08
25	30·89	33·63	35·58	34·58	35·56	35·58	34·84	33·27	37·01
30	27·81	30·34	32·17	31·21	31·66	32·66	31·33	30·92	33·96
35	24·92	27·09	29·03	28·05	28·17	29·43	27·61	28·36	30·73
40	22·19	23·75	25·21	24·66	24·95	26·40	24·24	25·49	27·30
45	19·50	20·71	22·57	21·61	21·33	23·35	20·62	22·34	23·77
50	17·07	17·72	19·26	18·46	17·98	20·40	17·49	19·41	20·24
55	14·77	14·98	16·15	15·53	14·83	17·47	14·14	16·72	16·88
60	12·31	12·24	13·08	12·63	12·09	14·86	11·84	14·10	13·86
65	9·86	9·78	10·49	10·10	9·57	12·30	9·26	11·56	11·07
70	7·45	7·66	7·91	7·72	7·01	10·00	7·52	9·13	8·34
75	5·51	5·89	6·03	5·91	5·53	7·87	5·36	6·81	5·79
80	4·07	4·27	4·47	4·28	4·27	5·75	4·24	5·05	4·73
85	2·37	3·16	3·40	3·23	2·83	3·90	3·21	3·38	3·45
90	·00	2·02	2·55	2·05	1·50	2·89	·00	2·47	1·79
95	·00	·00	1·07	1·00	·00	1·00	·00	·00	·83

TABLE III.

Showing the Number of Persons *living* and *dying* at every age, according to the observations of *M. De Parcieux*.

Age	Living	Dying	Age	Living	Dying	Age	Living	Dying
3	1000	30	34	702	8	65	395	15
4	970	22	35	694	8	66	380	16
5	948	18	36	686	8	67	364	17
6	930	15	37	678	7	68	347	18
7	915	13	38	671	7	69	329	19
8	902	12	39	664	7	70	310	19
9	890	10	40	657	7	71	291	20
10	880	8	41	650	7	72	271	20
11	872	6	42	643	7	73	251	20
12	866	6	43	636	7	74	231	20
13	860	6	44	629	7	75	211	19
14	854	6	45	622	7	76	192	19
15	848	6	46	615	8	77	173	19
16	842	7	47	607	8	78	154	18
17	835	7	48	599	9	79	136	18
18	828	7	49	590	9	80	118	17
19	821	7	50	581	10	81	101	16
20	814	8	51	571	11	82	85	14
21	806	8	52	560	11	83	71	12
22	798	8	53	549	11	84	59	11
23	790	8	54	538	12	85	48	10
24	782	8	55	526	12	86	38	9
25	774	8	56	514	12	87	29	7
26	766	8	57	502	13	88	22	6
27	758	8	58	489	13	89	16	5
28	750	8	59	476	13	90	11	4
29	742	8	60	463	13	91	7	3
30	734	8	61	450	13	92	4	2
31	726	8	62	437	14	93	2	1
32	718	8	63	423	14	94	1	1
33	710	8	64	409	14	95	0	0

TABLE IV.

Showing the *Expectation of Life*, according to the observations of *M. De Parcieux.*

Age	Expectation	Age	Expectation	Age	Expectation
3	47·71	34	31·52	65	11·26
4	48·17	35	30·88	66	10·69
5	48·27	36	30·23	67	10·14
6	48·20	37	29·58	68	9·61
7	47·98	38	28·89	69	9·11
8	47·66	39	28·18	70	8·64
9	47·30	40	27·48	71	8·17
10	46·83	41	26·77	72	7·73
11	46·26	42	26·06	73	7·31
12	45·58	43	25·34	74	6·90
13	44·89	44	24·62	75	6·50
14	44·20	45	23·89	76	6·10
15	43·51	46	23·15	77	5·71
16	42·82	47	22·45	78	5·36
17	42·17	48	21·74	79	5·00
18	41·52	49	21·07	80	4·69
19	40·87	50	20·38	81	4·39
20	40·22	51	19·73	82	4·01
21	39·62	52	19·11	83	3·84
22	39·00	53	18·48	84	3·52
23	38·40	54	17·85	85	3·21
24	37·78	55	17·25	86	2·92
25	37·17	56	16·04	87	2·07
26	36·55	57	16·02	88	2·36
27	35·93	58	15·44	89	2·06
28	35·30	59	14·84	90	1·77
29	34·69	60	14·25	91	1·50
30	34·06	61	13·65	92	1·25
31	33·29	62	13·04	93	1·00
32	32·80	63	12·43	94	·00
33	32·16	64	11·86		

TABLE V.

Showing the value of an annuity on a Single Life, deduced from the observations of *M. De Parcieux.*

Ages	3 per Cent	3½ per Cent	4 per Cent	4½ per Cent	5 per Cent	6 per Cent
3	22·028	19·959	18·242	16·756	15·475	13·391
4	22·390	20·319	18·559	17·052	15·752	13·633
5	22·597	20·518	18·749	17·233	15·923	13·787
6	22·726	20·647	18·877	17·357	16·043	13·897
7	22·791	20·720	18·953	17·435	16·121	13·972
8	22·813	20·751	18·996	17·482	16·171	14·024
9	22·815	20·770	19·022	17·515	16·209	14·066
10	22·766	20·742	19·008	17·512	16·213	14·079
11	22·664	20·665	18·949	17·408	16·179	14·061
12	22·506	20·539	18·844	17·380	16·105	14·007
13	22·343	20·403	18·734	17·289	16·029	13·951
14	22·175	20·266	18·620	17·194	15·949	13·892
15	22·002	20·123	18·502	17·095	15·865	13·830
16	21·823	19·976	18·380	16·991	15·777	13·765
17	21·666	19·849	18·275	16·905	15·705	13·713
18	21·505	19·717	18·167	16·815	15·629	13·658
19	21·339	19·581	18·054	16·721	15·551	13·601
20	21·168	19·441	17·938	16·624	15·469	13·541
21	21·020	19·321	17·841	16·544	15·403	13·496
22	20·867	19·197	17·740	16·462	15·336	13·450
23	20·711	19·071	17·637	16·377	15·265	13·401
24	20·550	18·940	17·530	16·289	15·193	13·350
25	20·386	18·805	17·420	16·198	15·117	13·298
26	20·217	13·667	17·306	16·104	15·039	13·243
27	20·043	18·524	17·188	16·006	14·957	13·185
28	19·864	18·377	17·066	15·905	14·873	13·126
29	19·681	18·225	16·940	15·800	14·785	13·063
30	19·492	18·069	16·810	15·691	14·693	12·998
31	19·298	17·907	16·675	15·578	14·598	12·930
32	19·099	17·741	16·535	15·460	14·499	12·858
33	18·893	17·568	16·390	15·338	14·395	12·783
34	18·682	17·390	16·240	15·211	14·287	12·704
35	18·464	17·207	16·084	15·078	14·175	12·022

TABLE V Continued.

Ages	3 per Cent	3½ per Cent	4 per Cent	4½ per Cent	5 per Cent	6 per Cent
36	18·240	17·017	15·922	14·941	14·057	12·535
37	18·009	16·829	15·755	14·797	13·934	12·441
38	17·742	16·590	15·556	14·624	13·783	12·329
39	17·407	16·352	15·349	14·444	13·625	12·206
40	17·183	16·105	15·133	14·254	13·459	12·076
41	16·889	15·848	14·907	14·056	13·284	11·939
42	16·585	15·581	14·673	13·849	13·100	11·793
43	16·271	15·304	14·427	13·631	12·906	11·638
44	15·946	15·016	14·171	13·403	12·702	11·473
45	15·609	14·716	13·904	13·164	12·487	11·299
46	15·200	14·405	13·625	12·913	12·261	11·113
47	14·925	14·105	13·357	12·672	12·044	10·935
48	14·576	13·794	13·076	12·419	11·815	10·746
49	14·244	13·494	12·807	12·170	11·595	10·564
50	13·909	13·183	12·526	11·921	11·363	10·372
51	13·567	12·883	12·255	11·675	11·140	10·187
52	13·248	12·596	11·995	11·440	10·927	10·010
53	12·919	12·298	11·725	11·195	10·703	9·823
54	12·579	11·989	11·443	10·938	10·468	9·625
55	12·252	11·692	11·173	10·691	10·242	9·436
56	11·914	11·383	10·891	10·433	10·006	9·235
57	11·565	11·066	10·597	10·163	9·757	9·024
58	11·228	10·757	10·314	9·892	9·517	8·819
59	10·881	10·436	10·020	9·631	9·266	8·604
60	10·522	10·101	9·715	9·346	9·003	8·370
61	10·151	9·760	9·393	9·049	8·726	8·135
62	9·766	9·402	9·060	8·753	8·435	7·880
63	9·392	9·053	8·734	8·433	8·150	7·629
64	9·005	8·691	8·391	8·114	7·850	7·363
65	8·604	8·314	8·049	7·780	7·535	7·082
66	8·212	7·914	7·691	7·451	7·224	6·803
67	7·830	7·584	7·350	7·129	6·918	6·528
68	7·460	7·234	7·019	6·814	6·629	6·259
69	7·104	6·896	6·699	6·511	6·331	5·997
70	6·766	6·575	6·391	6·221	6·055	5·747
71	6·424	6·250	6·084	5·925	5·773	5·489
72	6·105	5·946	5·794	5·648	5·505	5·248
73	5·789	5·641	5·506	5·373	5·246	5·006
74	5·479	5·348	5·222	5·101	4·985	4·766
75	5·178	5·060	4·945	4·836	4·730	4·53[illegible]

TABLE V Continued.

Ages	3 per Cent	3½ per Cent	4 per Cent	4½ per Cent	5 per Cent	6 per Cent
76	4·862	4·755	4·652	4·553	4·458	4·278
77	4·557	4·462	4·370	4·281	4·195	4·033
78	4·273	4·188	4·105	4·025	3·948	3·802
79	3·984	3·908	3·834	3·763	3·694	3·563
80	3·730	3·662	3·596	3·533	3·471	3·353
81	3·448	3·428	3·370	3·313	3·258	3·153
82	3·269	3·216	3·164	3·114	3·065	2·971
83	3·031	2·984	2·939	2·895	2·853	2·770
84	2·757	2·717	2·679	2·641	2·604	2·534
85	2·490	2·457	2·424	2·392	2·361	2·301
86	2·240	2·212	2·185	2·158	2·132	2·081
87	2·023	2·000	1·977	1·955	1·933	1·891
88	1·747	1·728	1·711	1·693	1·675	1·642
89	1·474	1·460	1·446	1·433	1·418	1·394
90	1·208	1·198	1·187	1·178	1·166	1·149
91	·955	·948	·941	·934	·924	·913
92	·721	·716	·712	·707	·703	·694
93	·485	·483	·481	·478	·476	·472
94	·000	·000	·000	·000	·000	·000

TABLE VI.

Showing the value of an annuity on Two Joint Lives, deduced from the observations of *M. De Parcieux.*

Difference of age 0 years.

Common Age	3½ per Cent	4½ per Cent	Common Age	3½ per Cent	4½ per Cent	Common Age	3½ per Cent	4½ per Cent
			36	13·839	12·408	71	3·968	3·817
			37	13·663	12·274	72	3·735	3·599
3	15·785	13·571	38	13·438	12·005	73	3·507	3·384
4	16·364	14·072	39	13·203	11·907	74	3·285	3·176
5	16·732	14·390	40	12·958	11·710	75	3·075	2·977
6	16·994	14·632	41	12·702	11·502	76	2·844	2·758
7	17·171	14·795	42	12·434	11·282	77	2·625	2·549
8	17·287	14·910	43	12·154	11·051	78	2·429	2·362
9	17·378	15·004	44	11·861	10·807	79	2·224	2·165
10	17·398	15·038	45	11·554	10·549	80	2·057	2·005
11	17·338	15·004	46	11·232	10·276	81	1·906	1·860
12	17·105	14·897	47	10·934	10·023	82	1·786	1·745
13	17·040	14·786	48	10·621	9·756	83	1·649	1·614
14	16·891	14·669	49	10·331	9·508	84	1·472	1·442
15	16·731	14·547	50	10·025	9·246	85	1·301	1·276
16	16·564	14·410	51	9·743	9·004	86	1·149	1·128
17	16·432	14·321	52	9·485	8·752	87	1·042	1·024
18	16·296	14·220	53	9·214	8·540	88	·873	·860
19	16·155	14·114	54	8·950	8·303	89	·709	·669
20	16·010	14·004	55	8·669	8·077	90	·552	·545
21	15·901	13·926	56	8·397	7·839	91	·410	·405
22	15·789	13·846	57	8·111	7·588	92	·300	·296
23	15·674	13·764	58	7·847	7·357	93	·242	·239
24	15·556	13·679	59	7·572	7·114	94	·000	·000
25	15·435	13·591	60	7·283	6·857			
26	15·311	13·501	61	6·979	6·585			
27	15·183	13·408	62	6·660	6·297			
28	15·051	13·312	63	6·357	6·024			
29	14·916	13·213	64	6·057	5·733			
30	14·776	13·110	65	5·700	5·423			
31	14·632	13·004	66	5·374	5·123			
32	14·484	12·893	67	5·062	4·835			
33	14·331	12·779	68	4·765	4·560			
34	14·172	12·660	69	4·486	4·300			
35	14·008	12·536	70	4·230	4·062			

TABLE VII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations of *M. De Parcieux.* Difference of age 5 years.

Ages	3½ per Cent	4½ per Cent	Ages	3½ per Cent	4½ per Cent	Ages	3½ per Ct	4½ per Ct.
			31–36	14·199	12·680	61–66	6·065	5·757
			32–37	14·035	12·556	62–67	5·749	5·467
3– 8	16·510	14·219	33–38	13·843	12·408	63–68	5·448	5·191
4– 9	16·854	14·525	34–39	13·644	12·252	64–69	5·151	4·917
5–10	17·052	14·708	35–40	13·436	12·089	65–70	4·858	4·647
6–11	17·155	14·811	36–41	13·220	11·918	66–71	4·568	4·377
7–12	17·172	14·840	37–42	12·995	11·739	67–72	4·300	4·128
8–13	17·154	14·841	38–43	12·740	11·531	68–73	4·040	3·885
9–14	17·121	14·828	39–44	12·473	11·313	69–74	3·792	3·653
10–15	17·048	14·783	40–45	12·194	11·082	70–75	3·561	3·436
11–16	16·934	14·701	41–46	11·902	10·839	71–76	3·314	3·203
12–17	16·796	14·598	42–47	11·617	10·601	72–77	3·088	2·989
13–18	16·653	14·491	43–48	11·318	10·349	73–78	2·877	2·788
14–19	16·505	14·380	44–49	11·025	10·102	74–79	2·663	2·585
15–20	16·351	14·264	45–50	10·718	9·841	75–80	2·478	2·409
16–21	16·213	14·161	46–51	10·416	9·583	76–81	2·293	2·232
17–22	16·091	14·071	47–52	10·137	9·345	77–82	2·130	2·076
18–23	15·965	13·979	48–53	9·845	9·095	78–83	1·964	1·917
19–24	15·835	13·884	49–54	9·557	8·846	79–84	1·770	1·730
20–25	15·701	13·784	50–55	9·274	8·602	80–85	1·596	1·562
21–26	15·583	13·699	51–56	8·994	8·360	81–86	1·437	1·408
22–27	15·462	13·612	52–57	8·719	8·120	82–87	1·316	1·292
23–28	15·338	13·522	53–58	8·449	7·886	83–88	1·150	1·130
24–29	15·210	13·429	54–59	8·168	7·639	84–89	·969	·954
25–30	15·079	13·333	55–60	7·889	7·394	85–90	·793	·782
26–31	14·913	13·233	56–61	7·597	7·136	86–91	·630	·621
27–32	14·804	13·131	57–62	7·291	6·862	87–92	·495	·489
28–33	14·660	13·024	58–63	7·003	6·605	88–93	·351	·348
29–34	14·511	12·914	59–64	6·701	6·333	89–94	·000	·000
30–35	14·358	12·799	60–65	6·383	6·045			

TABLE VIII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations of *M. De Parcieux.* Difference of age 10 years.

Ages	3½ per Cent	4½ per Cent	Ages	3½ per Cent	4½ per Cent	Ages	3½ per Ct.	4½ per Ct.
			31–41	13·500	12·133	61–71	5·054	4·828
			32–42	13·282	11·960	62–72	4·784	4·578
3–13	16·366	13·535	33–43	13·055	11·778	63–73	4·523	4·337
4–14	16·586	14·345	34–44	12·818	11·587	64–74	4·261	4·093
5–15	16·689	14·446	35–45	12·571	11·385	65–75	3·999	3·848
6–16	16·733	14·498	36–46	12·312	11·174	66–76	3·728	3·594
7–17	16·750	14·528	37–47	12·063	10·970	67–77	3·471	3·351
8–18	16·734	14·531	38–48	11·784	10·738	68–78	3·233	3·126
9–19	16·703	14·521	39–49	11·513	10·512	69–79	2·996	2·902
10–20	16·635	14·479	40–50	11·230	10·274	70–80	2·793	2·709
11–21	16·547	14·421	41–51	10·954	10·042	71–81	2·598	2·524
12–22	16·418	14·326	42–52	10·686	9·817	72–82	2·431	2·365
13–23	16·285	14·228	43–53	10·406	9·579	73–83	2·253	2·195
14–24	16·147	14·126	44–54	10·112	9·329	74–84	2·049	1·999
15–25	16·004	14·020	45–55	9·825	9·083	75–85	1·853	1·811
16–26	15·856	13·909	46–56	9·525	8·824	76–86	1·663	1·627
17–27	15·724	13·811	47–57	9·227	8·566	77–87	1·503	1·473
18–28	15·586	13·710	48–58	8·935	8·312	78–88	1·303	1·279
19–29	15·445	13·605	49–59	8·645	8·060	79–89	1·100	1·082
20–30	15·299	13·496	50–60	8·342	7·793	80–90	·909	·895
21–31	15·168	13·400	51–61	8·039	7·526	81–91	·727	·717
22–32	15·032	13·301	52–62	7·736	7·257	82–92	·566	·559
23–33	14·893	13·199	53–63	7·437	6·992	83–93	·401	·398
24–34	14·750	13·092	54–64	7·124	6·711	84–94	·000	·000
25–35	14·601	12·982	55–65	6·809	6·428			
26–36	14·448	12·868	56–66	6·496	6·145			
27–37	14·290	12·749	57–67	6·187	5·864			
28–38	14·104	12·606	58–68	5·895	5·598			
29–39	13·911	12·455	59–69	5·611	5·339			
30–40	13·710	12·298	60–70	5·337	5·088			

TABLE IX.

Showing the value of an annuity on Two Joint Lives, deduced from the observations of *M. De Parcieux.*

Difference of age 20 years.

Ages	3½ per Cent	4½ per Cent	Ages	3½ per Cent	4½ per Cent	Ages	3½ per Ct.	4½ per Ct.
			31–51	11·398	10·407	61–81	3·039	2·944
			32–52	11·163	10·212	62–82	2·849	2·764
3–23	15·560	13·566	33–53	10·918	10·008	63–83	2·647	2·573
4–24	15·773	13·765	34–54	10·663	9·794	64–84	2·409	2·346
5–25	15·876	13·870	35–55	10·418	9·589	65–85	2·173	2·120
6–26	15·925	13·929	36–56	10·163	9·374	66–86	1·954	1·909
7–27	15·929	13·951	37–57	9·897	9·148	67–87	1·766	1·729
8–28	15·903	13·946	38–58	9·625	8·916	68–88	1·528	1·498
9–29	15·861	13·929	39–59	9·342	8·673	69–89	1·293	1·270
10–30	15·784	13·882	40–60	9·046	8·417	70–90	1·066	1·049
11–31	15·668	13·801	41–61	8·737	8·147	71–91	·846	·834
12–32	15·511	13·684	42–62	8·413	7·862	72–92	·646	·638
13–33	15·348	13·562	43–63	8·095	7·582	73–93	·445	·440
14–34	15·179	13·434	44–64	7·762	7·285	74–94	·000	·000
15–35	15·004	13·301	45–65	7·412	6·972			
16–36	14·822	13·162	46–66	7·065	6·659			
17–37	14·652	13·033	47–67	6·734	6·360			
18–38	14·452	12·878	48–68	6·409	6·065			
19–39	14·245	12·716	49–69	6·103	5·787			
20–40	14·028	12·545	50–70	5·807	5·517			
21–41	13·821	12·382	51–71	5·515	5·250			
22–42	13·606	12·211	52–72	5·250	5·007			
23–43	13·381	12·032	53–73	4·984	4·762			
24–44	13·147	11·843	54–74	4·720	4·518			
25–45	12·902	11·645	55–75	4·470	4·286			
26–46	12·647	11·436	56–76	4·203	4·037			
27–47	12·402	11·236	57–77	3·943	3·794			
28–48	12·146	11·025	58–78	3·706	3·573			
29–49	11·901	10·823	59–79	3·463	3·343			
30–50	11·645	10·611	60–80	3·246	3·140			

TABLE X.

Showing the value of an annuity on Two Joint Lives, deduced from the observations of *M. De Parcieux.* Difference of age 30 years.

Ages	3½ per Cent	4½ per Cent	Ages	3½ per Cent.	4½ per Cent
3–33	14·605	12·893	36–66	7·357	6·920
4–34	14·761	13·048	37–67	7·043	6·639
5–35	14·813	13·112	38–68	6·726	6·353
6–36	14·810	13·130	39–69	6·420	6·076
7–37	14·764	13·111	40–70	6·127	5·811
8–38	14·662	13·043	41–71	5·828	5·538
9–39	14·542	12·960	42–72	5·548	5·282
10–40	14·384	12·843	43–73	5·268	5·025
11–41	14·186	12·690	44–74	4·990	4·770
12–42	13·946	12·498	45–75	4·718	4·518
13–43	13·694	12·296	46–76	4·427	4·248
14–44	13·432	12·084	47–77	4·152	3·991
15–45	13·158	11·860	48–78	3·892	3·748
16–46	12·872	11·624	49–79	3·632	3·503
17–47	12·611	11·410	50–80	3·399	3·284
18–48	12·339	11·185	51–81	3·182	3·080
19–49	12·076	10·968	52–82	2·991	2·899
20–50	11·801	10·739	53–83	2·780	2·700
21–51	11·551	10·532	54–84	2·533	2·464
22–52	11·313	10·335	55–85	2·296	2·238
23–53	11·064	10·128	56–86	2·072	2·023
24–54	10·805	9·911	57–87	1·877	1·836
25–55	10·557	9·702	58–88	1·629	1·596
26–56	10·298	9·484	59–89	1·381	1·356
27–57	10·029	9·255	60–90	1·138	1·119
28–58	9·769	9·034	61–91	·904	·891
29–59	9·499	8·803	62–92	·636	·677
30–60	9·218	8·561	63–93	·467	·463
31–61	8·924	8·306	64–94	·000	·000
32–62	8·617	8·038			
33–63	8·318	7·775			
34–64	8·005	7·499			
35–65	7·678	7·207			

TABLE XI.

Showing the value of an annuity on Two Joint Lives, deduced from the observations of *M. De Parcieux.*
Difference of age 40 years.

Ages	3½ per Cent	4½ per Cent	Ages	3½ per Cent	4½ per Cent
3–43	12·996	11·665	31–71	5·874	5·578
4–44	13·021	11·706	32–72	5·601	5·329
5–45	12·944	11·658	33–73	5·329	5·081
6–46	12·812	11·560	34–74	5·062	4·835
7–47	12·656	11·440	35–75	4·802	4·595
8–48	12·465	11·289	36–76	4·525	4·338
9–49	12·275	11·138	37–77	4·259	4·091
10–50	12·048	10·954	38–78	4·004	3·852
11–51	11·804	10·754	39–79	3·742	3·607
12–52	11·543	10·538	40–80	3·511	3·390
13–53	11·272	10·311	41–81	3·292	3·184
14–54	10·989	10·073	42–82	3·092	2·996
15–55	10·715	9·843	43–83	2·874	2·790
16–56	10·430	9·601	44–84	2·619	2·547
17–57	10·145	9·359	45–85	2·370	2·309
18–58	9·871	9·125	46–86	2·133	2·082
19–59	9·585	8·879	47–87	1·931	1·888
20–60	9·287	8·621	48–88	1·670	1·636
21–61	8·987	8·362	49–89	1·413	1·387
22–62	8·675	8·088	50–90	1·160	1·141
23–63	8·369	7·820	51–91	·919	·906
24–64	8·050	7·538	52–92	·698	·689
25–65	7·717	7·241	53–93	·473	·469
26–66	7·389	6·948	54–94	·000	·000
27–67	7·068	6·659			
28–68	6·755	6·378			
29–69	6·454	6·105			
30–70	6·166	5·845			

TABLE XII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations of *M. De Parcieux.* Difference of age 50 years.

Ages	$3\frac{1}{2}$ per Cent	$4\frac{1}{2}$ per Cent	Ages	$3\frac{1}{2}$ per Cent	$4\frac{1}{2}$ per Cent
3–53	10·680	9·770	26–76	4·529	4·342
4–54	10·629	9·741	27–77	4·258	4·089
5–55	10·513	9 653	28–78	4·003	3·851
6–56	10·351	9·522	29–79	3·742	3·607
7–57	10·149	9·356	30–80	3·513	3·391
8–58	9·939	9·181	31–81	3·294	3·186
9–59	9·710	8·980	32–82	3·097	3·000
10–60	9·449	8·767	33–83	2·880	2·796
11–61	9·155	8·513	34–84	2·628	2·556
12–62	8·825	8·224	35–85	2·382	2·320
13–63	8·502	7·941	36–86	2·150	2·099
14–64	8·164	7·643	37–87	1·951	1·908
15–65	7·812	7·328	38–88	1·689	1·655
16–66	7·464	7·017	39–89	1·430	1·403
17–67	7·132	6·719	40–90	1·175	1·156
18–68	6·809	6·428	41–91	·932	·918
19–69	6·496	6·145	42–92	·706	·697
20–70	6·197	5·873	43–93	·478	·473
21–71	5·901	5·603	44–94	·000	·000
22–72	5·624	5·351			
23–73	5·348	5·098			
24–74	5·076	4·848			
25–75	4·811	4·604			

TABLE XIII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations of *M. De Parcieux.*

Difference of age 60 years.

Ages	3½ per Cent	4½ per Cent
3–63	8·049	7·520
4–64	7·883	7·379
5–65	7·644	7·169
6–66	7·383	6·938
7–67	7·108	6·693
8–68	6·828	6·443
9–69	6·554	6·197
10–70	6·281	5·951
11–71	5·989	5·685
12–72	5·702	5·424
13–73	5·417	5·162
14–74	5·134	4·903
15–75	4·859	4·649
16–76	4·566	4·377
17–77	4·289	4·119
18–78	4·029	3·876
19–79	3·762	3·625
20–80	3·526	3·404
21–81	3·306	3·197
22–82	3·107	3·010
23–83	2·889	2·804
24–84	2·635	2·562
25–85	2·386	2·325
26–86	2·152	2·100
27–87	1·950	1·907
28–88	1·689	1·654
29–89	1·429	1·403
30–90	1·175	1·155
31–91	·931	·918
32–92	·706	·697
33–93	·478	·473
34–94	·000	·000

TABLE XIV.

Showing the Number of Persons *living* and *dying* at every age, according to the observations made in *Sweden*.

Age	Males		Females		Lives in general	
	Living	Dying	Living	Dying	Living	Dying
0	10000	2300	10000	2090	10000	2195
1	7700	500	7910	518	7805	509
2	7200	337	7392	350	7296	344
3	6863	240	7042	250	6952	245
4	6623	150	6792	135	6707	143
5	6473	125	6657	120	6564	122
6	6348	105	6537	105	6442	105
7	6243	90	6432	85	6337	87
8	6153	75	6347	70	6250	73
9	6078	65	6277	60	6177	62
10	6013	55	6217	52	6115	54
11	5958	45	6165	46	6061	45
12	5913	45	6119	40	6016	42
13	5868	40	6079	35	5974	38
14	5828	40	6044	35	5936	37
15	5788	39	6009	35	5899	37
16	5749	39	5974	40	5862	40
17	5710	39	5934	40	5822	40
18	5671	44	5894	42	5782	42
19	5627	44	5852	43	5740	43
20	5583	50	5809	43	5697	47
21	5533	50	5766	43	5650	47
22	5483	50	5723	43	5603	48
23	5433	55	5680	44	5555	48
24	5378	55	5636	45	5507	50
25	5323	55	5591	45	5457	50
26	5268	55	5546	50	5407	52
27	5213	55	5496	52	5355	54
28	5158	55	5444	55	5301	55
29	5103	56	5389	55	5246	55
30	5049	59	5334	60	5191	59

TABLE XIV Continued.

Ages	Males		Females		Lives in general	
	Living	Dying	Living	Dying	Living	Dying
31	4988	60	5274	60	5132	60
32	4928	60	5214	65	5072	62
33	4868	60	5149	65	5010	63
34	4808	60	5084	65	4947	63
35	4748	60	5019	60	4884	59
36	4688	60	4959	56	4825	58
37	4628	60	4903	56	4767	58
38	4568	60	4847	56	4709	58
39	4508	60	4791	58	4651	60
40	4448	65	4733	65	4591	65
41	4383	72	4668	75	4526	73
42	4311	80	4593	76	4453	78
43	4231	80	4517	76	4375	78
44	4151	80	4441	75	4297	78
45	4071	80	4366	72	4219	76
46	3991	80	4294	67	4143	74
47	3911	80	4227	65	4069	72
48	3831	80	4162	65	3997	73
49	3751	85	4097	70	3924	78
50	3666	95	4027	75	3846	85
51	3571	95	3952	80	3761	87
52	3476	95	3872	85	3674	90
53	3381	95	3787	85	3584	90
54	3286	95	3702	85	3494	91
55	3191	95	3617	85	3403	91
56	3096	95	3532	85	3312	92
57	3001	100	3447	90	3220	95
58	2901	100	3357	90	3125	95
59	2801	100	3267	100	3030	100
60	2701	105	3167	110	2930	108
61	2596	110	3057	118	2822	114
62	2486	115	2939	120	2708	118
63	2371	115	2819	120	2590	118
64	2256	115	2699	120	2472	118
65	2141	115	2579	120	2354	118
66	2026	115	2459	120	2236	118
67	1911	120	2339	120	2118	121
68	1791	125	2219	120	1997	124
69	1666	125	2099	120	1873	124
70	1541	125	1979	130	1749	127

TABLE XIV Continued.

Ages	Males		Females		Lives in general	
	Living	Dying	Living	Dying	Living	Dying
71	1416	125	1849	140	1622	133
72	1291	120	1709	150	1489	135
73	1171	120	1559	160	1354	140
74	1051	110	1399	150	1214	130
75	941	105	1249	140	1084	121
76	836	100	1109	130	963	115
77	736	90	979	120	848	105
78	646	85	859	110	743	95
79	561	80	749	100	648	90
80	481	75	649	95	558	90
81	406	70	554	90	468	84
82	336	65	464	85	384	75
83	271	60	379	80	309	65
84	211	50	299	75	244	55
85	161	40	224	55	189	45
86	121	30	169	40	144	35
87	91	22	129	30	109	27
88	69	17	99	23	82	20
89	52	14	76	18	62	15
90	38	12	58	15	47	14
91	26	9	43	12	33	12
92	17	7	31	10	21	10
93	10	6	21	8	11	6
94	4	3	13	6	5	3
95	1	1	7	4	2	1
96	0	0	3	2	1	1
97	0	0	1	1	0	0

TABLE XV.

Showing the *Expectation of Life*, deduced from the observations made in *Sweden*.

Ages	Males	Females	Lives in general	Ages	Males	Females	Lives in general
1	42·45	44·00	42·95	31	29·69	31·54	30·57
2	43·83	46·05	44·92	32	29·04	30·91	29·94
3	44·96	47·31	46·11	33	28·39	30·28	29·30
4	45·57	48·04	46·78	34	27·74	29·66	28·67
5	45·62	48·00	46·79	35	27·09	29·03	28·03
6	45·50	47·87	46·66	36	26·43	28·26	27·31
7	45·26	47·64	46·43	37	25·76	27·50	26·68
8	44·91	47·28	46·07	38	25·09	26·74	26·01
9	44·46	46·80	45·61	39	24·42	25·97	25·33
10	43·94	46·25	45·07	40	23·75	25·21	24·66
11	43·26	45·55	44·38	41	23·15	24·68	24·05
12	42·58	44·85	43·70	42	22·54	24·75	23·44
13	41·91	44·15	43·01	43	21·93	23·62	22·83
14	41·24	43·46	42·33	44	21·32	23·10	22·22
15	40·56	42·76	41·64	45	20·71	22·57	21·61
16	39·83	42·04	40·92	46	20·12	21·91	20·98
17	39·11	41·31	40·19	47	19·52	21·24	20·35
18	38·39	40·59	39·47	48	18·92	20·58	19·72
19	37·67	39·87	38·74	49	18·32	19·92	19·09
20	36·95	39·15	38·02	50	17·72	19·26	18·46
21	36·28	38·43	37·33	51	17·17	18·64	17·87
22	35·62	37·72	36·64	52	16·63	18·01	17·29
23	34·96	37·01	35·96	53	16·08	17·39	16·70
24	34·30	36·29	35·27	54	15·53	16·77	16·12
25	33·63	35·58	34·58	55	14·98	16·15	15·53
26	32·98	34·90	33·91	56	14·43	15·53	14·95
27	32·32	34·21	33·23	57	13·87	14·92	14·37
28	31·66	33·53	32·56	58	13·33	14·31	13·79
29	31·00	32·85	31·88	59	12·79	13·69	13·21
30	30·34	32·17	31·21	60	12·24	13·08	12·63

TABLE XV Continued.

Ages	Males	Female	Lives in general	Ages	Males	Female	Lives in general
61	11·72	12·56	12·12	81	3·96	4·13	4·01
62	11·21	12·04	11·62	82	3·69	3·84	3·80
63	10·73	11·52	11·11	83	3·45	3·59	3·57
64	10·26	11·01	10·61	84	3·30	3·42	3·39
65	9·78	10·49	10·10	85	3·16	3·40	3·23
66	9·30	9·97	9·62	86	3·04	3·34	3·09
67	8·84	9·46	9·15	87	2·88	3·22	2·92
68	8·40	8·94	8·67	88	2·64	3·05	2·71
69	7·99	8·42	8·20	89	2·34	2·82	2·43
70	7·60	7·91	7·72	90	2·02	2·55	2·05
71	7·22	7·53	7·32	91	1·73	2·27	1·71
72	6·87	7·16	6·89	92	1·38	1·95	1·40
73	6·53	6·78	6·53	93	1·00	1·64	1·23
74	6·22	6·40	6·23	94	·75	1·35	1·10
75	5·89	6·03	5·91	95	·00	1·07	1·00
76	5·56	5·73	5·59	96	·00	·83	·00
77	5·25	5·43	5·28	97	·00	·00	·00
78	4·92	5·11	4·96				
79	4·59	4·79	4·61				
80	4·27	4·47	4·28				

TABLE XVI.

Showing the value of an annuity on a Single Life, at every age (Male or Female), deduced from the observations made in *Sweden.*

Ages	Males		Females	
	4 per Cent	5 per Cent	4 per Cent	5 per Cent
1	16·503	14·051	16·820	14·271
2	17·355	14·778	17·719	15·034
3	17·935	15·279	18·344	15·571
4	18·328	15·624	18·780	15·951
5	18·503	15·786	18·927	16·088
6	18·622	15·901	19·045	16·203
7	18·693	15·977	19·131	16·291
8	18·725	16·021	19·162	16·335
9	18·715	16·030	19·151	16·343
10	18·674	16·014	19·109	16·325
11	18·600	15·970	19·041	16·286
12	18·491	15·896	18·952	16·229
13	18·378	15·819	18·840	16·153
14	18·246	15·724	18·707	16·059
15	18·105	15·624	18·568	15·960
16	17·958	15·517	18·424	15·856
17	17·803	15·404	18·290	15·761
18	17·643	15·285	18·151	15·662
19	17·492	15·175	18·013	15·563
20	17·335	15·059	17·872	15·462
21	17·192	14·955	17·725	15·356
22	17·042	14·846	17·573	15·245
23	16·887	14·732	17·414	15·129
24	16·742	14·627	17·252	15·039
25	16·592	14·517	17·087	14·886
26	16·436	14·402	16·915	14·757
27	16·274	14·282	16·751	14·636
28	16·105	14·156	16·588	14·515
29	15·930	14·024	16·427	14·396
30	15·751	13·889	16·261	14·272

TABLE XVI Continued.

Ages	Males		Females	
	4 per Cent	5 per Cent	4 per Cent	5 per Cent
31	15·575	13·756	16·104	14·156
32	15·395	13·619	15·941	14·035
33	15·208	13·477	15·787	13·923
34	15·014	13·327	15·629	13·806
35	14·812	13·170	15·465	13·684
36	14·601	13·006	15·278	13·542
37	14·382	12·833	15·070	13·382
38	14·154	12·652	14·854	13·2 3
39	13·916	12·462	14·629	13·036
40	13·668	12·261	14·401	12·856
41	13·426	12·065	14·185	12·687
42	13·196	11·880	13·994	12·538
43	12·984	11·710	13·798	12·387
44	12·763	11·532	13·596	12·229
45	12·535	11·347	13·383	12·061
46	12·297	11·153	13·151	11·876
47	12·051	10·951	12·894	11·668
48	11·795	10·738	12·630	11·443
49	11·528	10·516	12·333	11·205
50	11·267	10·298	12·049	10·970
51	11·030	10·100	11·769	10·737
52	10·785	9·895	11·492	10·507
53	10·531	9·682	11·220	10·280
54	10·269	9·460	10·957	10·042
55	9·998	9·229	10·642	9·792
56	9·717	8·988	10·334	9·529
57	9·425	8·736	10·012	9·253
58	9·140	8·489	9·692	8·976
59	8·845	8·232	9·358	8· 84
60	8·540	7·963	9·039	8·406
61	8·241	7·700	8·739	8·144
62	7·950	7·442	8·453	7·895
63	7·669	7·193	8·166	7·643
64	7·382	6·938	7·870	7·382
65	7·090	6·676	7·556	7·111

TABLE XVI Continued.

Ages	Males		Females	
	4 per Cent	5 per Cent	4 per Cent	5 per Cent
66	6·792	6·408	7·252	6·831
67	6·489	6·134	6·930	6·541
68	6·201	5·872	6·596	6·239
69	5·933	5·628	6·253	5·926
70	5·670	5·389	5·897	5·599
71	5·418	5·158	5·564	5·293
72	5·180	4·940	5·261	5·013
73	4·940	4·719	4·998	4·770
74	4·724	4·521	4·792	4·581
75	4·487	4·302	4·582	4·388
76	4·253	4·084	4·367	4·189
77	4·024	3·871	4·145	3·983
78	3·768	3·631	3·913	3·767
79	3·512	3·390	3·668	3·536
80	3·260	3·152	3·402	3·285
81	3·017	2·921	3·145	3·041
82	2·792	2·706	2·905	2·812
83	2·600	2·523	2·699	2·615
84	2·473	2·403	2·559	2·480
85	2·371	2·306	2·552	2·476
86	2·281	2·222	2·518	2·446
87	2·154	2·103	2·431	2·365
88	1·955	1·912	2·294	2·236
89	1·698	1·664	2·108	2·059
90	1·417	1·392	1·873	1·833
91	1·154	1·136	1·628	1·596
92	·835	·824	1·349	1·325
93	·477	·471	1·071	1·054
94	·240	·238	·799	·788
95	·000	·000	·544	·537
96	·000	·000	·320	·317

TABLE XVII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations (among Males and Females collectively) made in *Sweden*.

Difference of age 0 years.

Common Age	4 per Cent	Common Age	4 per Cent	Common Age	4 per Cent	Common Age	4 per Cent
1	12·252	26	13·671	51	8·469	76	2·490
2	13·583	27	13·495	52	8·230	77	2·340
3	14·558	28	13·323	53	7·994	78	2·170
4	15·267	29	13·148	54	7·748	79	1·967
5	15·577	30	12·965	55	7·495	80	1·758
6	15·820	31	12·795	56	7·229	81	1·600
7	16·003	32	12·624	57	6·954	82	1·472
8	16·109	33	12·456	58	6·678	83	1·364
9	16·152	34	12·286	59	6·388	84	1·276
10	16·141	35	12·109	60	6·104	85	1·212
11	16·087	36	11·904	61	5·844	86	1·172
12	15·982	37	11·683	62	5·600	87	1·127
13	15·855	38	11·452	63	5·367	88	1·071
14	15·701	39	11·209	64	5·128	89	·949
15	15·535	40	10·964	65	4·881	90	·718
16	15·361	41	10·732	66	4·626	91	·516
17	15·196	42	10·531	67	4·362	92	·326
18	15·023	43	10·346	68	4·103	93	·236
19	14·854	44	10·154	69	3·851	94	·190
20	14·682	45	9·954	70	3·593	95	·024
21	14·525	46	9·736	71	3·345	96	·000
22	14·360	47	9·497	72	3·128		
23	14·194	48	9·236	73	2·935		
24	14·020	49	8·966	74	2·797		
25	13·849	50	8·707	75	2·648		

TABLE XVIII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations (among Males and Females collectively) made in *Sweden*.

Difference of age 6 years.

Ages	4 per Cent	Ages	4 per Cent	Ages	4 per Cent
1– 7	13·989	31–37	12·192	61–67	4·984
2– 8	14·780	32–38	11·988	62–68	4·729
3– 9	15·323	33–39	11·779	63–69	4·482
4–10	15·665	34–40	11·568	64–70	4·231
5–11	15·817	35–41	11·361	65–71	3·982
6–12	15·887	36–42	11·156	66–72	3·750
7–13	15·914	37–43	10·953	67–73	3·527
8–14	15·888	38–44	10·741	68–74	3·340
9–15	15·824	39–45	10·519	69–75	3·147
10–16	15·729	40–46	10·286	70–76	2·946
11–17	15·617	41–47	10·049	71–77	2·752
12–18	15·477	42–48	9·813	72–78	2·558
13–19	15·327	43–49	9·581	73–79	2·355
14–20	15·164	44–50	9·351	74–80	2·172
15–21	15·001	45–51	9·129	75–81	2·017
16–22	14·832	46–52	8·897	76–82	1·877
17–23	14·665	47–53	8·658	77–83	1·756
18–24	14·491	48–54	8·402	78–84	1·629
19–25	14·320	49–55	8·139	79–85	1·524
20–26	14·144	50–56	7·874	80–86	1·416
21–27	13·976	51–57	7·613	81–87	1·320
22–28	13·807	52–58	7·351	82–88	1·225
23–29	13·635	53–59	7·083	83–89	1·094
24–30	13·455	54–60	6·814	84–90	·902
25–31	13·284	55–61	6·555	85–91	·725
26–32	13·108	56–62	6·299	86–92	·556
27–33	12·935	57–63	6·045	87–93	·459
28–34	12·763	58–64	5·788	88–94	·396
29–35	12·586	59–65	5·519	89–95	·364
30–36	12·390	60–66	5·249	90–96	·000

TABLE XIX.

Showing the value of an annuity on Two Joint Lives, deduced from the observations (among Males and Females collectively) made in *Sweden*.

Difference of age 12 years.

Ages	4 per Cent	Ages	4 per Cent	Ages	4 per Cent
1–13	13·894	31–43	11·359	61–73	3·927
2–14	14·557	32–44	11·170	62–74	3·747
3–15	14·988	33–45	10·978	63–75	3·563
4–16	15·259	34–46	10·775	64–76	3·370
5–17	15·326	35–47	10·557	65–77	3·180
6–18	15·354	36–48	10·314	66–78	2·974
7–19	15·351	37–49	10·059	67–79	2·743
8–20	15·310	38–50	9·805	68–80	2·514
9–21	15·244	39–51	9·558	69–81	2·324
10–22	15·149	40–52	9·308	70–82	2·155
11–23	15·033	41–53	9·066	71–83	2·004
12–24	14·889	42–54	8·830	72–84	1·875
13–25	14·736	43–55	8·597	73–85	1·768
14–26	14·566	44–56	8·354	74–86	1·692
15–27	14·392	45–57	8·101	75–87	1·605
16–28	14·216	46–58	7·841	76–88	1·497
17–29	14·042	47–59	7·563	77–89	1·339
18–30	13·8[illegible]0	48–60	7·281	78–90	1·097
19–31	13·687	49–61	7·003	79–91	·803
20–32	13·512	50–62	6·749	80–92	·638
21–33	13·345	51–63	6·505	81–93	·511
22–34	13·173	52–64	6·256	82–94	·427
23–35	12·997	53–65	6·004	83–95	·379
24–36	12·801	54–66	5·743	84–96	·000
25–37	12·599	55–67	5·474		
26–38	12·387	56–68	5·204		
27–39	12·170	57–69	4·936		
28–40	11·953	58–70	4·664		
29–41	11·742	59–71	4·395		
30–42	11·543	60–72	4·149		

TABLE XX.

Showing the value of an annuity on Two Joint Lives, deduced from the observations (among Males and Females collectively) made in *Sweden.*

Difference of age 18 years.

Ages	4 per Cent	Ages	4 per Cent	Ages	4 per Cent
1–19	13·389	31–49	10·365	61–79	2·974
2–20	14·008	32–50	10·128	62–80	2·744
3–21	14·417	33–51	9·905	63–81	2·557
4–22	14·671	34–52	9·679	64–82	2·396
5–23	14·725	35–53	9·452	65–83	2·252
6–24	14·740	36–54	9·207	66–84	2·123
7–25	14·727	37–55	8·951	67–85	2·010
8–26	14·673	38–56	8·683	68–86	1·910
9–27	14·590	39–57	8·404	69–87	1·798
10–28	14·484	40–58	8·124	70–88	1·661
11–29	14·357	41–59	7·839	71–89	1·464
12–30	14·202	42–60	7·569	72–90	1·189
13–31	14·045	43–61	7·318	73–91	·937
14–32	13·874	44–62	7·075	74–92	·708
15–33	13·700	45–63	6·836	75–93	·575
16–34	13·520	46–64	6·586	76–94	·481
17–35	13·340	47–65	6·323	77–95	·421
18–36	13·141	48–66	6·048	78–96	·000
19–37	12·934	49–67	5·764		
20–38	12·720	50–68	5·487		
21–39	12·505	51–69	5·221		
22–40	12·296	52–70	4·953		
23–41	12·073	53–71	4·694		
24–42	11·873	54–72	4·455		
25–43	11·683	55–73	4·231		
26–44	11·485	56–74	4·043		
27–45	11·284	57–75	3·844		
28–46	11·072	58–76	3·637		
29–47	10·847	59–77	3·430		
30–48	10·606	60–78	3·210		

TABLE XXI.

Showing the value of an annuity on Two Joint Lives, deduced from the observations (among Males and Females collectively) made in *Sweden*.

Difference of age 24 years.

Ages	4 per Cent	Ages	4 per Cent	Ages	4 per Cent
1–25	12·832	26–50	10·364	51–75	4·008
2–26	13·409	27–51	10·130	52–76	3·803
3–27	13·778	28–52	9·864	53–77	3·605
4–28	14·003	29–53	9·659	54–78	3·389
5–29	14·037	30–54	9·413	55–79	3·150
6–30	14·033	31–55	9·167	56–80	2·909
7–31	14·006	32–56	8·912	57–81	2·710
8–32	13·944	33–57	8·651	58–82	2·539
9–33	13·855	34–58	8·389	59–83	2·385
10–34	13·741	35–59	8·114	60–84	2·248
11–35	13·604	36–60	7·833	61–85	2·135
12–36	13·428	37–61	7·561	62–86	2·037
13–37	13·234	38–62	7·296	63–87	1·926
14–38	13·023	39–63	7·033	64–88	1·790
15–39	12·798	40–64	6·763	65–89	1·585
16–40	12·570	41–65	6·492	66–90	1·290
17–41	12·351	42–66	6·225	67–91	1·017
18–42	12·146	43–67	5·957	68–92	·764
19–43	11·951	44–68	5·689	69–93	·617
20–44	11·751	45–69	5·426	70–94	·514
21–45	11·550	46–70	5·153	71–95	·441
22–46	11·335	47–71	4·884	72–96	·000
23–47	11·107	48–72	4·633		
24–48	10·862	49–73	4·398		
25–49	10·612	50–74	4·205		

TABLE XXII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations (among Males and Females collectively) made in *Sweden*.

Difference of age 30 years.

Ages	4 per Cent	Ages	4 per Cent	Ages	4 per Cent
1–31	12·196	26–56	9·080	51–81	2·792
2–32	12·730	27–57	8·807	52–82	2·623
3–33	13·066	28–58	8·534	53–83	2·475
4–34	13·264	29–59	8·250	54–84	2·344
5–35	13·277	30–60	7·967	55–85	2·232
6–36	13·242	31–61	7·702	56–86	2·130
7–37	13·170	32–62	7·446	57–87	2·010
8–38	13·059	33–63	7·196	58–88	1·864
9–39	12·913	34–64	6·942	59–89	1·644
10–40	12·743	35–65	6·679	60–90	1·333
11–41	12·563	36–66	6·402	61–91	1·050
12–42	12·379	37–67	6·115	62–92	·789
13–43	12·196	38–68	5·828	63–93	·639
14–44	11·997	39–69	5·543	64–94	·533
15–45	11·787	40–70	5·254	65–95	·456
16–46	11·562	41–71	4·977	66–96	·000
17–47	11·328	42–72	4·730		
18–48	11·076	43–73	4·507		
19–49	10·819	44–74	4·322		
20–50	10·567	45–75	4·128		
21–51	10·332	46–76	3·921		
22–52	10·092	47–77	3·715		
23–53	9·852	48–78	3·489		
24–54	9·602	49–79	3·238		
25–55	9·347	50–80	2·990		

TABLE XXIII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations (among Males and Females collectively) made in *Sweden*.

Difference of age 36 years.

Ages	4 per Cent	Ages	4 per Cent
1–37	11·465	31–67	6·197
2–38	11·913	32–68	5·917
3–39	12·164	33–69	5·642
4–40	12·284	34–70	5·364
5–41	12·242	35–71	5·093
6–42	12·185	36–72	4·840
7–43	12·112	37–73	4·603
8–44	12·004	38–74	4·405
9–45	11·865	39–75	4·195
10–46	11·694	40–76	3·975
11–47	11·493	41–77	3·762
12–48	11·259	42–78	3·539
13–49	11·011	43–79	3·295
14–50	10·759	44–80	3·052
15–51	10·514	45–81	2·854
16–52	10·264	46–82	2·684
17–53	10·018	47–83	2·533
18–54	9·761	48–84	2·396
19–55	9·500	49–85	2·277
20–56	9·228	50–86	2·171
21–57	8·953	51–87	2·050
22–58	8·675	52–88	1·901
23–59	8·385	53–89	1·681
24–60	8·097	54–90	1·366
25–61	7·823	55–91	1·078
26–62	7·557	56–92	·810
27–63	7·297	57–93	·655
28–64	7·032	58–94	·546
29–65	6·761	59–95	·464
30–66	6·481	60–96	·000

TABLE XXIV.

Showing the value of an annuity on Two Joint Lives, deduced from the observations (among Males and Females collectively) made in *Sweden*.

Difference of age 42 years.

Ages	4 per Cent	Ages	4 per Cent
1–43	10·546	31–73	4·646
2–44	10·946	32–74	4·453
3–45	11·168	33–75	4·251
4–46	11·200	34–76	4·040
5–47	11·183	35–77	3·833
6–48	11·064	36–78	3·605
7–49	10·915	37–79	3·352
8–50	10·743	38–80	3·098
9–51	10·560	39–81	2·889
10–52	10·357	40–82	2·710
11–53	10·140	41–83	2·553
12–54	9·898	42–84	2·418
13–55	9·641	43–85	2·305
14–56	9·371	44–86	2·203
15–57	9·087	45–87	2·083
16–58	8·799	46–88	1·933
17–59	8·503	47–89	1·708
18–60	8·208	48–90	1·385
19–61	7·928	49–91	1·090
20–62	7·658	50–92	·818
21–63	7·396	51–93	·662
22–64	7·127	52–94	·551
23–65	6·851	53–95	·468
24–66	6·566	54–96	·000
25–67	6·275		
26–68	5·986		
27–69	5·702		
28–70	5·415		
29–71	5·136		
30–72	4·881		

TABLE XXV.

Showing the number of persons *living* and *dying* at every age, according to the observations made at *Northampton.*

Age	Living	Dying	Age	Living	Dying	Ages	Living	Dying
0	11650	1340						
1	8650	1367	36	3935	75	71	1152	80
2	7283	502	37	3860	75	72	1072	80
3	6781	335	38	3785	75	73	992	80
4	6446	197	39	3710	75	74	912	80
5	6249	184	40	3635	76	75	832	80
6	6065	140	41	3559	77	76	752	77
7	5925	110	42	3482	78	77	675	73
8	5815	80	43	3404	78	78	602	68
9	5735	60	44	3326	78	79	534	65
10	5675	52	45	3248	78	80	469	63
11	5623	50	46	3170	78	81	406	60
12	5573	50	47	3092	78	82	346	57
13	5523	50	48	3014	78	83	289	55
14	5473	50	49	2936	79	84	234	48
15	5423	50	50	2857	81	85	186	41
16	5373	53	51	2776	82	86	145	34
17	5320	58	52	2694	82	87	111	28
18	5262	63	53	2612	82	88	83	21
19	5199	67	54	2530	82	89	62	16
20	5132	72	55	2448	82	90	46	12
21	5060	75	56	2366	82	91	34	10
22	4985	75	57	2284	82	92	24	8
23	4910	75	58	2202	82	93	16	7
24	4835	75	59	2120	82	94	9	5
25	4760	75	60	2038	82	95	4	3
26	4685	75	61	1956	82	96	1	1
27	4610	75	62	1874	81	97	0	0
28	4535	75	63	1793	81			
29	4460	75	64	1712	80			
30	4385	75	65	1632	80			
31	4310	75	66	1552	80			
32	4235	75	67	1472	80			
33	4160	75	68	1392	80			
34	4085	75	69	1312	80			
35	4010	75	70	1232	80			

TABLE XXVI.

Showing the *Expectations of Life* at every age, deduced from the observations made at *Northampton.*

Ages	Expectat.	Ages	Expectat.	Ages	Expectat.	Ages	Expectat.
1	32·74	26	30·33	51	17·50	76	6·18
2	37·79	27	29·82	52	17·02	77	5·83
3	39·55	28	29·30	53	16·54	78	5·48
4	40·58	29	28·79	54	16·06	79	5·11
5	40·84	30	28·27	55	15·58	80	4·75
6	41·07	31	27·76	56	15·10	81	4·41
7	41·03	32	27·24	57	14·63	82	4·09
8	40·79	33	26·72	58	14·15	83	3·80
9	40·36	34	26·20	59	13·68	84	3·58
10	39·78	35	25·68	60	13·21	85	3·37
11	39·14	36	25·16	61	12·75	86	3·19
12	38·49	37	24·64	62	12·28	87	3·01
13	37·83	38	24·12	63	11·81	88	2·86
14	37·17	39	23·60	64	11·35	89	2·66
15	36·51	40	23·08	65	10·88	90	2·41
16	35·85	41	22·56	66	10·42	91	2·09
17	35·20	42	22·04	67	9·96	92	1·75
18	34·58	43	21·54	68	9·50	93	1·37
19	33·99	44	21·03	69	9·05	94	1·05
20	33·43	45	20·52	70	8·60	95	·75
21	32·90	46	20·02	71	8·17	96	·50
22	32·39	47	19·51	72	7·74	97	·00
23	31·88	48	19·00	73	7·33		
24	31·36	49	18·49	74	6·92		
25	30·85	50	17·99	75	6·54		

TABLE XXVII.

Showing the value of an annuity on a Single Life, at every age, deduced from the observations made at *Northampton*.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent	7 per Cent	8 per Cent
1	16·021	13·465	11·563	10·107	8·963	8·046
2	18·599	15·633	13·420	11·724	10·391	9·321
3	19·575	16·462	14·135	12·348	10·941	9·812
4	20·210	17·010	14·613	12·769	11·315	10·147
5	20·473	17·248	14·827	12·962	11·489	10·304
6	20·727	17·482	15·041	13·156	11·666	10·466
7	20·853	17·611	15·166	13·275	11·777	10·570
8	20·885	17·662	15·226	13·337	11·840	10·631
9	20·812	17·625	15·210	13·335	11·846	10·641
10	20·663	17·523	15·139	13·285	11·809	10·614
11	20·480	17·393	15·043	13·212	11·752	10·560
12	20·283	17·251	14·937	13·130	11·687	10·517
13	20·081	17·103	14·826	13·044	11·618	10·461
14	19·872	16·950	14·710	12·953	11·545	10·401
15	19·657	16·791	14·588	12·857	11·467	10·337
16	19·435	16·625	14·460	12·755	11·384	10·268
17	19·218	16·462	14·334	12·655	11·302	10·200
18	19·013	16·309	14·217	12·562	11·226	10·137
19	18·820	16·167	14·108	12·477	11·157	10·081
20	18·638	16·033	14·007	12·398	11·094	10·030
21	18·470	15·912	13·917	12·329	11·042	9·986
22	18·311	15·797	13·833	12·265	10·993	9·947
23	18·148	15·680	13·746	12·200	10·942	9·907
24	17·983	15·560	13·658	12·132	10·890	9·865
25	17·814	15·438	13·567	12·063	10·836	9·823
26	17·642	15·312	13·473	11·992	10·780	9·778
27	17·467	15·184	13·377	11·917	10·723	9·732
28	17·289	15·053	13·278	11·841	10·663	9·685
29	17·107	14·918	13·177	11·763	10·602	9·635
30	16·922	14·781	13·072	11·682	10·539	9·584

TABLE XXVII Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent	7 per Cent	8 per Cent
31	16·732	14·639	12·965	11·598	10·473	9·531
32	16·540	14·495	12·854	11·512	10·404	9·476
33	16·343	14·347	12·740	11·423	10·333	9·418
34	16·142	14·195	12·623	11·331	10·260	9·359
35	15·938	14·039	12·502	11·236	10·183	9·296
36	15·729	13·880	12·377	11·137	10·104	9·231
37	15·515	13·716	12·249	11·035	10·021	9·164
38	15·298	13·548	12·116	10·929	9·935	9·093
39	15·075	13·375	11·979	10·819	9·845	9·019
40	14·848	13·197	11·837	10·705	9·752	8·941
41	14·620	13·018	11·695	10·589	9·657	8·863
42	14·391	12·838	11·551	10·473	9·562	8·783
43	14·162	12·657	11·407	10·356	9·466	8·703
44	13·929	12·472	11·258	10·235	9·366	8·620
45	13·692	12·283	11·105	10·110	9·262	8·533
46	13·450	12·089	10·947	9·980	9·154	8·443
47	13·203	11·890	10·784	9·846	9·042	8·348
48	12·951	11·685	10·616	9·707	8·925	8·249
49	12·693	11·475	10·443	9·563	8·804	8·146
50	12·436	11·264	10·269	9·417	8·681	8·041
51	12·183	11·057	10·097	9·273	8·559	7·937
52	11·930	10·849	9·925	9·129	8·437	7·833
53	11·674	10·637	9·748	8·980	8·311	7·725
54	11·414	10·421	9·567	8·827	8·181	7·614
55	11·150	10·201	9·382	8·670	8·047	7·499
56	10·882	9·977	9·193	8·509	7·909	7·379
57	10·611	9·749	8·999	8·343	7·766	7·256
58	10·337	9·516	8·801	8·173	7·619	7·128
59	10·058	9·280	8·599	7·999	7·468	6·996
60	9·777	9·039	8·392	7·820	7·312	6·860
61	9·493	8·795	8·181	7·637	7·152	6·719
62	9·205	8·547	7·966	7·449	6·988	6·574
63	8·910	8·291	7·742	7·253	6·815	6·421
64	8·611	8·030	7·514	7·052	6·637	6·262
65	8·304	7·761	7·276	6·841	6·449	6·095

TABLE XXVII Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent	7 per Cent	8 per Cent
66	7·994	7·488	7·034	6·625	6·256	5·922
67	7·682	7·211	6·787	6·405	6·058	5·743
68	7·367	6·930	6·536	6·179	5·855	5·559
69	7·051	6·647	6·281	5·949	5·646	5·370
70	6·734	6·361	6·023	5·716	5·434	5·176
71	6·418	6·075	5·764	5·479	5·218	4·978
72	6·103	5·790	5·504	5·241	5·000	4·778
73	5·794	5·507	5·245	5·004	4·781	4·576
74	5·491	5·230	4·990	4·769	4·565	4·375
75	5·199	4·962	4·744	4·542	4·354	4·180
76	4·925	4·710	4·511	4·326	4·154	3·994
77	4·652	4·457	4·277	4·109	3·952	3·806
78	4·372	4·197	4·035	3·884	3·742	3·609
79	4·077	3·921	3·776	3·641	3·514	3·394
80	3·781	3·643	3·515	3·394	3·281	3·174
81	3·499	3·377	3·263	3·156	3·055	2·960
82	3·229	3·122	3·020	2·926	2·836	2·751
83	2·982	2·887	2·797	2·713	2·632	2·557
84	2·793	2·708	2·627	2·551	2·479	2·410
85	2·620	2·543	2·471	2·402	2·337	2·275
86	2·462	2·393	2·328	2·266	2·207	2·151
87	2·312	2·251	2·193	2·138	2·085	2·035
88	2·185	2·131	2·080	2·031	1·984	1·939
89	2·013	1·967	1·924	1·882	1·842	1·803
90	1·794	1·758	1·723	1·689	1·656	1·625
91	1·501	1·474	1·447	1·422	1·398	1·374
92	1·190	1·171	1·153	1·136	1·118	1·102
93	·839	·827	·816	·806	·795	·785
94	·536	·530	·524	·518	·512	·507
95	·242	·240	·238	·236	·234	·232
96	·000	·000	·000	·000	·000	·000

TABLE XXVIII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton.*

Difference of age 0 years.

Common Age	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1	9·491	8·252	7·287	6·515
2	12·789	11·107	9·793	8·741
3	14·196	12·325	10·862	9·689
4	15·181	13·185	11·621	10·365
5	15·638	13·591	11·984	10·691
6	16·099	14·005	12·358	11·031
7	16·375	14·224	12·596	11·251
8	16·510	14·399	12·731	11·382
9	16·483	14·396	12·744	11·404
10	16·339	14·277	12·665	11·345
11	16·142	14·133	12·546	11·249
12	15·926	13·966	12·411	11·139
13	15·702	13·789	12·268	11·023
14	15·470	13·604	12·118	10·899
15	15·229	13·411	11·960	10·767
16	14·979	13·212	11·793	10·626
17	14·737	13·019	11·630	10·489
18	14·516	12·841	11·483	10·365
19	14·316	12·679	11·351	10·255
20	14·133	12·535	11·232	10·156
21	13 974	12·409	11·131	10·074
22	13·830	12·293	11·042	10·002
23	13·683	12·179	10·951	9·928
24	13·534	12·062	10·858	9·853
25	13·383	11·944	10·764	9·776
26	13·230	11·822	10·667	9·697
27	13·074	11 699	10·567	9·616
28	12·915	11·573	10·466	9·533
29	12·754	11·445	10·362	9·448
30	12·589	11·313	10·255	9·360

TABLE XXVIII Continued.

Common Age	3 per Cent.	4 per Cent	5 per Cent	6 per Cent
31	12·422	11·179	10·146	9·270
32	12·252	11·042	10·034	9·178
33	12·079	10·902	9·919	9·082
34	11·902	10·759	9·801	8·984
35	11·722	10·612	9·680	8·883
36	11·539	10·462	9·555	8·778
37	11·351	10·307	9·427	8·670
38	11·160	10·149	9·294	8·558
39	10·964	9·986	9·158	8·442
40	10·764	9·820	9·016	8·322
41	10·565	9·654	8·876	8·202
42	10·369	9·491	8·737	8·083
43	10·175	9·326	8·599	7·965
44	9·978	9·160	8·457	7·843
45	9·776	8·990	8·312	7·718
46	9·571	8·815	8·162	7·589
47	9·362	8·637	8·008	7·455
48	9·149	8·453	7·849	7·316
49	8·931	8·266	7·686	7·173
50	8·714	8·081	7·522	7·030
51	8·507	7·900	7·366	6·893
52	8·304	7·724	7·213	6·758
53	8·099	7·544	7·056	6·620
54	7·891	7·362	6·897	6·480
55	7·681	7·179	6·735	6·336
56	7·470	6·993	6·571	6·190
57	7·256	6·805	6·404	6·041
58	7·041	6·614	6·234	5·890
59	6·824	6·421	6·062	5·735
60	6·606	6·226	5·888	5·579
61	6·387	6·030	5·712	5·420
62	6·166	5·831	5·533	5·259
63	5·938	5·626	5·347	5·089
64	5·709	5·417	5·158	4·917
65	5·471	5·201	4·960	4·736

TABLE XXVIII Continued.

Common Age	3 per Cent	4 per Cent	5 per Cent	6 per Cent
66	5·231	4·982	4·759	4·551
67	4·990	4·760	4·555	4·363
68	4·747	4·537	4·348	4·171
69	4·504	4·312	4·140	3·977
70	4·261	4·087	3·930	3·781
71	4·020	3·862	3·719	3·584
72	3·781	3·639	3·510	3·387
73	3·548	3·421	3·304	3·193
74	3·324	3·211	3·105	3·005
75	3·114	3·015	2·917	2·827
76	2·926	2·833	2·750	2·668
77	2·741	2·656	2·583	2·511
78	2·550	2·470	2·410	2·346
79	2·338	2·271	2·217	2·161
80	2·122	2·068	2·018	1·969
81	1·917	1·869	1·827	1·786
82	1·719	1·681	1·642	1·606
83	1·538	1·510	1·472	1·441
84	1·416	1·387	1·357	1·330
85	1·309	1·339	1·256	1·232
86	1·218	1·195	1·171	1·149
87	1·141	1·124	1·098	1·078
88	1·103	1·030	1·063	1·044
89	1·036	1·015	1·001	·984
90	·938	·922	·909	·895
91	·769	·756	·748	·737
92	·591	·583	·576	·569
93	·369	·365	·361	·357
94	·203	·201	·199	·197
95	·060	·060	·059	·058
96	·000	·000	·000	·000

TABLE XXIX.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton.*

Difference of age 5 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1– 6	12·347	10·741	9·479	8·467
2– 7	14·461	12·581	11·100	9·911
3– 8	15·300	13·319	11·755	10·498
4– 9	15·809	13·775	12·165	10·869
5–10	15·974	13·933	12·315	11·010
6–11	16·110	14·068	12·447	11·136
7–12	16·137	14·111	12·498	11·192
8–13	16·089	14·089	12·492	11·197
9–14	15·957	13·992	12·421	11·144
10–15	15·762	13·841	12·302	11·048
11–16	15·538	13·664	12·158	10·929
12–17	15·308	13·480	12·009	10·805
13–18	15·086	13·303	11·864	10·685
14–19	14·870	13·130	11·723	10·568
15–20	14·660	12·961	11·585	10·453
16–21	14·457	12·799	11·452	10·342
17–22	14·265	12·646	11·327	10·239
18–23	14·082	12·500	11·209	10·140
19–24	13·908	12·361	11·096	10·048
20–25	13·741	12·229	10·989	9·960
21–26	13·584	12·105	10·890	9·879
22–27	13·433	11·987	10·796	9·803
23–28	13·280	11·866	10·699	9·724
24–29	13·124	11·743	10·600	9·643
25–30	12·966	11·618	10·499	9·561
26–31	12·805	11·489	10·396	9·476
27–32	12·641	11·359	10·289	9·389
28–33	12·474	11·225	10·181	9·299
29–34	12·304	11·088	10·069	9·207
30–35	12·131	10·948	9·954	9·112

TABLE XXIX Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
31–36	11·955	10·805	9·837	9·014
32–37	11·775	10·659	9·716	8·913
33–38	11·592	10·508	9·591	8·808
34–39	11·404	10·354	9·463	8·701
35–40	11·213	10·196	9·331	8·589
36–41	11·021	10·037	9·198	8·476
37–42	10·828	9·877	9·062	8·362
38–43	10·635	9·716	8·927	8·246
39–44	10·437	9·550	8·787	8·127
40–45	10·236	9·381	8·643	8·003
41–46	10·033	9·210	8·497	7·878
42–47	9·829	9·037	8·350	7·751
43–48	9·624	8·862	8·200	7·621
44–49	9·414	8·683	8·046	7·488
45–50	9·204	8·503	7·891	7·353
46–51	8·997	8·326	7·737	7·219
47–52	8·790	8·147	7·582	7·084
48–53	8·579	7·965	7·424	6·945
49–54	8·366	7·780	7·262	6·802
50–55	8·152	7·593	7·098	6·658
51–56	7·941	7·409	6·936	6·515
52–57	7·730	7·225	6·774	6·371
53–58	7·518	7·039	6·609	6·225
54–59	7·304	6·850	6·442	6·076
55–60	7·088	6·659	6·272	5·924
56–61	6·870	6·465	6·100	5·770
57–62	6·651	6·270	5·925	5·613
58–63	6·427	6·070	5·744	5·450
59–64	6·201	5·867	5·561	5·284
60–65	5·970	5·658	5·372	5·112
61–66	5·737	5·447	5·180	4·938
62–67	5·503	5·285	4·986	4·760
63–68	5·265	5·017	4·786	4·576
64–69	5·025	4·798	4·585	4·390
65–70	4·783	4·573	4·378	4·199

TABLE XXIX Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
66–71	4·540	4·349	4·169	4·005
67–72	4·298	4·124	3·960	3·811
68–73	4·059	3·901	3·752	3·616
69–74	3·825	3·683	3·547	3·423
70–75	3·599	3·471	3·347	3·236
71–76	3·386	3·270	3·159	3·059
72–77	3·176	3·070	2·971	2·882
73–78	2·963	[illegible]	2·780	2·701
74–79	2·743	2·659	2·580	2·511
75–80	2·526	2·448	2·381	2·323
76–81	2·325	2·258	2·195	2·147
77–82	2·131	2·077	2·013	1·975
78–83	1·947	1·899	1·838	1·810
79–84	1·793	1·751	1·750	1·672
80–85	1·645	1·608	1·573	1·539
81–86	1·511	1·478	1·447	1·417
82–87	1·385	1·356	1·329	1·303
83–88	1·284	1·250	1·235	[illegible]
84–89	1·188	[illegible]	1·145	1·124
85–90	1·074	1·054	1·038	1·021
86–91	·921	[illegible]	·892	·879
87–92	·756	[illegible]	·734	·725
88–93	[illegible]	[illegible]	·547	·541
89–94	·377	[illegible]	·359	·365
90–95	·179	·177	·175	·174
91–96	·000	·000	·000	·000

TABLE XXX.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton.*

Difference of age 10 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1–11	12·346	10·782	9·544	8·547
2–12	14·239	12·438	11·010	9·857
3–13	14·895	13·019	11·528	10·324
4–14	15·287	13·374	11·850	10·617
5–15	15·391	13·479	11·954	10·716
6–16	15·486	13·578	12·052	10·812
7–17	15·490	13·599	12·083	10·849
8–18	15·436	13·569	12·070	10·847
9–19	15·316	13·482	12·006	10·799
10–20	15·151	13·355	11·906	10·719
11–21	14·974	13·217	11·797	10·631
12–22	14·795	13·078	11·686	10·541
13–23	14·612	12·934	11·570	10·446
14–24	14·424	12·784	11·450	10·348
15–25	14·230	12·630	11·324	10·244
16–26	14·030	12·470	11·193	10·135
17–27	13·832	12·311	11·063	10·027
18–28	13·642	12·158	10·939	9·924
19–29	13·461	12·013	10·820	9·826
20–30	13·286	11·873	10·707	9·732
21–31	13·121	11·742	10·600	9·644
22–32	12·961	11·615	10·498	9·561
23–33	12·798	11·485	10·393	9·474
24–34	12·632	11·352	10·285	9·386
25–35	12·463	11·217	10·175	9·295
26–36	12·291	11·078	10·062	9·201
27–37	12·116	10·936	9·946	9·105
28–38	11·937	10·791	9·826	9·005
29–39	11·755	10·642	9·703	8·902
30–40	11·568	10·490	9·576	8·795

TABLE XXX Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
31–41	11·382	10·336	9·448	8·688
32–42	11·195	10·182	9·320	8·580
33–43	11·007	10·027	9·190	8·471
34–44	10·817	9·869	9·058	8·358
35–45	10·622	9·706	8·921	8·242
36–46	10·424	9·540	8·781	8·122
37–47	10·221	9·370	8·636	7·998
38–48	10·014	9·195	8·487	7·870
39–49	9·803	9·015	8·333	7·737
40–50	9·590	8·834	8·177	7·602
41–51	9·383	8·658	8·025	7·470
42–52	9·179	8·483	7·875	7·340
43–53	8·975	8·308	7·724	7·208
44–54	8·767	8·130	7·569	7·073
45–55	8·557	7·948	7·411	6·935
46–56	8·344	7·763	7·249	6·793
47–57	8·127	7·574	7·084	6·648
48–58	7·907	7·382	6·915	6·498
49–59	7·684	7·186	6·742	6·344
50–60	7·461	6·989	6·568	6·189
51–61	7·240	6·795	6·395	6·035
52–62	7·021	6·600	6·222	5·880
53–63	6·795	6·399	6·042	5·719
54–64	6·568	6·196	5·860	5·555
55–65	6·334	5·986	5·671	5·384
56–66	6·098	5·774	5·479	5·209
57–67	5·800	5·559	5·283	5·031
58–68	5·621	5·341	5·084	4·849
59–69	5·380	5·121	4·883	4·665
60–70	5·139	4·900	4·680	4·478
61–71	4·898	4·679	4·476	4·289
62–72	4·659	4·458	4·272	4·099
63–73	4·420	4·236	4·066	3·908
64–74	4·186	4·019	3·864	3·719
65–75	3·958	3·806	3·665	3·533

TABLE XXX Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
66–76	3·743	3.606	3·477	3·357
67–77	3·529	3·405	3·289	3·180
68–78	3·310	3·199	3·095	2·996
69–79	3·077	2·979	2·887	2·799
70–80	2·843	2·757	2·675	2·598
71–81	2·618	2·542	2·470	2·402
72–82	2·401	2·334	2·271	2·211
73–83	2·199	2·141	2·085	2·032
74–84	2·043	1·991	1·941	1·894
75–85	1·903	1·856	1·811	1·769
76–86	1·781	1·739	1·699	1·661
77–87	1·670	1·633	1·597	1·562
78–88	1·580	1·546	1·514	1·483
79–89	1·456	1·427	1·400	1·373
80–90	1·302	1·278	1·255	1·234
81–91	1·096	1·078	1·061	1·044
82–92	·877	·864	·852	·840
83–93	·622	·614	·606	·599
84–94	·408	·403	·398	·394
85–95	·189	·187	·185	·183
86–96	·000	·000	·000	·000

TABLE XXXI.

Showing the value of an annuity on Two Join Lives, deduced from the observations made a *Northampton.*

Difference of age 15 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 ·per Cent
1–16	11·864	10·406	9·243	8·301
2–17	13·659	11·981	10·612	9·555
3–18	14·277	12·531	11·134	9·908
4–19	14·557	12·876	11·447	10·284
5–20	14·776	12·993	11·561	10·391
6–21	14·904	13·121	11·635	10·510
7–22	14·950	13·178	11·748	10·576
8–23	14·929	13·178	11·751	10·597
9–24	14·834	13·112	11·715	10·566
10–25	14·683	12·998	11·627	10·497
11–26	14·508	12·861	11·519	10·410
12–27	14·323	12·715	11·402	10·314
13–28	14·132	12·564	11·280	10·215
14–29	13·926	12·408	11·153	10·110
15–30	13·734	12·246	11·021	10·001
16–31	13·527	12·078	10·883	9·886
17–32	13·320	11·911	10·746	9·771
18–33	13·121	11·750	10·613	9·660
19–34	12·930	11·595	10·486	9·554
20–35	12·744	11·445	10·363	9·451
21–36	12·567	11·302	10·246	9·354
22–37	12·394	11·163	10·132	9·260
23–38	12·218	11·020	10·015	9·163
24–39	12·038	10·874	9·895	9·063
25–40	11·854	10·725	9·771	8·960
26–41	11·670	10·574	9·647	8·855
27–42	11·486	10·423	9·522	8·751
28–43	11·302	10·272	9·306	8·645
29–44	11·114	10·117	9·267	8·536
30–45	10·923	9·959	9·135	8·424

TABLE XXXI Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
31–46	10·728	9·797	8·998	8·309
32–47	10·530	9·631	8·858	8·189
33–48	10·327	9·461	8·714	8·066
34–49	10·120	9·286	8·565	7·938
35–50	9·912	9·110	8·415	7·809
36–51	9·707	8·937	8·267	7·681
37–52	9·503	8·763	8·119	7·553
38–53	9·296	8·586	7·966	7·421
39–54	9·085	8·406	7·810	7·286
40–55	8·870	8·221	7·651	7·146
41–56	8·655	8·035	7·489	7·005
42–57	8·439	7·848	7·326	6·862
43–58	8·222	7·660	7·162	6·718
44–59	8·003	7·469	6·994	6·570
45–60	7·781	7·274	6·822	6·418
46–61	7·556	7·076	6·648	6·263
47–62	7·328	6·875	6·469	6·104
48–63	7·093	6·667	6·283	5·937
49–64	6·854	6·454	6·093	5·767
50–65	6·611	6·236	5·897	5·590
51–66	6·369	6·019	5·701	5·412
52–67	6·127	5·801	5·504	5·233
53–68	5·884	5·580	5·303	5·050
54–69	5·638	5·357	5·100	4·864
55–70	5·391	5·132	4·893	4·674
56–71	5·145	4·905	4·685	4·482
57–72	4·899	4·679	4·477	4·289
58–73	4·656	4·455	4·269	4·096
59–74	4·418	4·234	4·064	3·906
60–75	4·189	4·021	3·866	3·721
61–76	3·974	3·821	3·679	3·546
62–77	3·760	3·621	3·492	3·371
63–78	3·538	3·414	3·297	3·188
64–79	3·303	3·192	3·088	2·990
65–80	3·063	2·965	2·873	2·786

TABLE XXXI Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
66–81	2·833	2·746	2·664	2·587
67–82	2·610	2·533	2·461	2·393
68–83	2·403	2·336	2·272	2·211
69–84	2·244	2·183	2·126	2·071
70–85	2·097	2·042	1·991	1·941
71–86	1·963	1·914	1·867	1·823
72–87	1·838	1·794	1·753	1·713
73–88	1·736	1·697	1·660	1·625
74–89	1·603	1·570	1·538	1·508
75–90	1·440	1·413	1·387	1·361
76–91	1·221	1·200	1·180	1·160
77–92	·985	·970	·955	·942
78–93	·706	·697	·688	·679
79–94	·458	·453	·448	·443
80–95	·210	·208	·206	·204
81–96	·000	·000	·000	·000

TABLE XXXII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton.*

Difference of age 20 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1–21	11·413	10·053	8·961	8·070
2–22	13·172	11·605	10·344	9·313
3–23	13·794	12·161	10·843	9·704
4–24	14·178	12·511	11·163	10·057
5–25	14·301	12·633	11·281	10·170
6–26	14·420	12·754	11·400	10·285
7–27	14·451	12·798	11·452	10·341
8–28	14·417	12·786	11·455	10·354
9–29	14·310	12·710	11·401	10·315
10–30	14·150	12·586	11·304	10·239
11–31	13·965	12·441	11·168	10·144
12–32	13·770	12·285	11·062	10·042
13–33	13·570	12·125	10·932	9·934
14–34	13·363	11·959	10·796	9·822
15–35	13·151	11·787	10·655	9·703
16–36	12·932	11·609	10·507	9·579
17–37	12·714	11·430	10·358	9·454
18–38	12·502	11·257	10·214	9·333
19–39	12·297	11·089	10·074	9·215
20–40	12·096	10·924	9·957	9·100
21–41	11·906	10·768	9·809	8·992
22–42	11·723	10·619	9·685	8·889
23–43	11·540	10·470	9·562	8·785
24–44	11·354	10·317	9·435	8·670
25–45	11·164	10·160	9·304	8·559
26–46	10·970	10·000	9·170	8·455
27–47	10·773	9·836	9·032	8·338
28–48	10·572	9·667	8·890	8·217
29–49	10·366	9·495	8·744	8·092
30–50	10·160	9·321	8·596	7·966
31–51	9·957	9·151	8·451	7·841
32–52	9·755	8·980	8·306	7·716
33–53	9·550	8·806	8·157	7·588
34–54	9·342	8·629	8·005	7·457
35–55	9·131	8·448	7·849	7·322

TABLE XXXII Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
36–56	8·916	8·204	7·690	7·183
37–57	8·699	8·076	7·527	7·041
38–58	8·477	7·884	7·360	6·894
39–59	8·253	7·689	7·189	6·744
40–60	8·025	7·490	7·015	6·590
41–61	7·796	7·290	6·838	6·434
42–62	7·567	7·088	6·660	6·276
43–63	7·332	6·881	6·477	6·112
44–64	7·095	6·671	6·289	5·941
45–65	6·850	6·453	6·094	5·769
46–66	6·602	6·230	5·894	5·588
47–67	6·351	6·004	5·690	5·403
48–68	6·096	5·774	5·481	5·213
49–69	5·839	5·541	5·268	5·019
50–70	5·582	5·306	5·054	4·822
51–71	5·328	5·074	4·841	4·626
52–72	5·077	4·845	4·639	4·430
53–73	4·829	4·614	4·417	4·234
54–74	4·585	4·389	4·208	4·040
55–75	4·350	4·171	4·006	3·852
56–76	4·129	3·966	3·815	3·674
57–77	3·908	3·761	3·623	3·494
58–78	3·682	3·549	3·424	3·308
59–79	3·440	3·322	3·210	3·105
60–80	3·197	3·092	2·992	2·899
61–81	2·964	2·870	2·782	2·699
62–82	2·739	2·656	2·578	2·504
63–83	2·530	2·457	2·387	2·321
64–84	2·371	2·305	2·242	2·182
65–85	2·223	2·163	2·107	2·053
66–86	2·089	2·035	1·984	1·936
67–87	1·963	1·915	1·870	1·826
68–88	1·860	1·817	1·777	1·737
69–89	1·722	1·685	1·650	1·616
70–90	1·545	1·515	1·486	1·459
71–91	1·303	1·280	1·259	1·238
72–92	1·044	1·028	1·012	·997
73–93	·743	·733	·723	·714
74–94	·480	·474	·469	·464
75–95	·219	·217	·215	·213

TABLE XXXIII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton.*

Difference of age 25 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1–26	11·037	9·770	8·742	7·897
2–27	12·722	11·264	10·080	9·104
3–28	13·307	11·790	10·555	9·537
4–29	13·661	12·116	10·855	9·813
5–30	13·762	12·220	10·959	9·913
6–31	13·859	12·322	11·062	10·015
7–32	13·871	12·350	11·100	10·060
8–33	13·820	12·323	11·090	10 061
9–34	13·698	12·234	11·024	10·012
10–35	13·525	12·098	10·916	9·925
11–36	13·328	11·941	10·788	9·820
12–37	13·120	11·773	10·651	9·707
13–38	12·906	11·600	10·509	9·588
14–39	12·686	11·420	10·360	9·464
15–40	12·459	11·234	10·205	9·333
16–41	12·229	11·044	10·046	9·198
17–42	12·002	10·856	9·889	9·065
18–43	11·785	10·677	9·739	8·938
19–44	11·574	10·502	9·592	8·814
20–45	11·367	10·330	9·448	8·692
21–46	11·167	10·165	9·310	8·574
22–47	10·969	10·001	9·173	8·458
23–48	10·768	9·833	9·031	8·338
24–49	10·562	9·661	8·886	8·214
25–50	10 356	9·488	8·739	8·089
26–51	10·154	9·318	8·595	7·966
27–52	9·952	9·148	8·451	7·842
28–53	9·748	8·975	8·304	7·716
29–54	9·540	8·799	8·153	7·586
30–55	9·329	8·619	7·999	7·453

TABLE XXXIII Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
31–56	9·115	8·436	7·841	7·316
32–57	8·897	8·250	7·680	7·175
33–58	8·677	8·060	7·515	7·031
34–59	8·454	7·866	7·346	6·884
35–60	8·227	7·669	7·174	6·732
36–61	7·997	7·469	6·998	6·577
37–62	7·765	7·265	6·819	6·418
38–63	7·525	7·053	6·631	6·252
39–64	7·281	6·838	6·440	6·081
40–65	7·030	6·614	6·240	5·901
41–66	6·776	6·388	6·037	5·718
42–67	6·522	6·159	5·831	5·532
43–68	6·266	5·929	5·622	5·343
44–69	6·008	5·696	5·411	5·150
45–70	5·749	5·460	5·195	4·953
46–71	5·488	5·222	4·978	4·753
47–72	5·228	4·983	4·758	4·551
48–73	4·970	4·746	4·539	4·348
49–74	4·716	4·511	4·322	4·146
50–75	4·472	4·285	4·112	3·951
51–76	4·245	4·074	3·916	3·768
52–77	4·019	3·864	3·720	3·586
53–78	3·787	3·648	3·518	3·396
54–79	3·540	3·416	3·299	3·189
55–80	3·291	3·180	3·076	2·978
56–81	3·051	2·953	2·861	2·774
57–82	2·820	2·733	2·651	2·574
58–83	2·608	2·530	2·457	2·388
59–84	2·446	2·376	2·310	2·247
60–85	2·297	2·234	2·174	2·118
61–86	2·162	2·105	2·051	2·000
62–87	2·036	1·985	1·937	1·891
63–88	1·932	1·886	1·843	1·802
64–89	1·790	1·751	1·714	1·678
65–90	1·606	1·575	1·544	1·515
66–91	1·354	1·330	1·307	1·285
67–92	1·083	1·067	1·050	1·035
68–93	·770	·760	·750	·740
69–94	·497	·491	·485	·480
70–95	·227	·224	·222	·220
71–96	·000	·000	·000	·000

TABLE XXXIV.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton.*

Difference of age 30 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1–31	10·605	9·438	8·483	7·691
2–32	12·203	10·865	9·767	8·855
3–33	12·743	11·355	10·213	9·263
4–34	13·061	11·651	10·488	9·518
5–35	13·136	11·732	10·572	9·602
6–36	13·207	11·812	10·656	9·687
7–37	13·195	11·819	10·676	9·715
8–38	13·122	11·772	10·648	9·701
9–39	12·981	11·665	10·565	9·637
10–40	12·791	11·513	10·442	9·537
11–41	12·580	11·342	10·302	9·420
12–42	12·363	11·165	10·156	9·298
13–43	12·144	10·985	10·007	9·173
14–44	11·918	10·799	9·852	9·042
15–45	11·687	10·607	9·690	8·905
16–46	11·448	10·408	9·522	8·762
17–47	11·210	10·208	9·353	8·617
18–48	10·975	10·011	9·186	8·473
19–49	10·746	9·818	9·021	8·332
20–50	10·523	9·630	8·861	8·195
21–51	10·313	9·454	8·712	8·067
22–52	10·111	9·284	8·568	7·944
23–53	9·905	9·111	8·421	7·818
24–54	9·696	8·934	8·270	7·688
25–55	9·484	8·754	8·116	7·555
26–56	9·269	8·570	7·958	7·419
27–57	9·051	8·383	7·797	7·279
28–58	8·830	8·193	7·632	7·135
29–59	8·605	7·999	7·464	6·988
30–60	8·378	7·802	7·292	6·837

TABLE XXXIV Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
31-61	8·147	7·601	7·116	6·682
32-62	7·914	7·397	6·937	6·524
33-63	7·673	7·186	6·750	6·359
34-64	7·429	6·971	6·559	6·189
35-65	7·177	6·747	6·360	6·010
36-66	6·922	6·520	6·156	5·827
37-67	6·663	6·288	5·948	5·639
38-68	6·401	6·052	5·735	5·446
39-69	6·137	5·813	5·518	5·249
40-70	5·871	5·571	5·298	5·047
41-71	5·605	5·329	5·076	4·844
42-72	5·341	5·087	4·854	4·640
43-73	5·081	4·848	4·634	4·436
44-74	4·826	4·613	4·417	4·235
45-75	4·580	4·386	4·206	4·040
46-76	4·348	4·171	4·006	3·853
47-77	4·115	3·954	3·805	3·666
48-78	3·875	3·731	3·586	3·469
49-79	3·619	3·490	3·369	3·256
50-80	3·362	3·247	3·140	3·039
51-81	3·117	3·015	2·920	2·829
52-82	2·882	2·792	2·707	2·627
53-83	2·665	2·585	2·510	2·438
54-84	2·501	2·428	2·350	2·295
55-85	2·349	2·284	2·222	2·164
56-86	2·211	2·153	2·097	2·044
57-87	2·082	2·030	1·980	1·932
58-88	1·975	1·928	1·883	1·841
59-89	1·828	1·788	1·750	1·713
60-90	1·641	1·608	1·577	1·547
61-91	1·382	1·358	1·334	1·311
62-92	1·105	1·088	1·071	1·055
63-93	·785	·774	·764	·754
64-94	·506	·500	·494	·489
65-95	·230	·228	·226	·224
66-96	·000	·000	·000	·000

TABLE XXXV.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton.*

Difference of age 35 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1–36	10·104	9·047	8·173	7·442
2–37	11·600	10·392	9·390	8·551
3–38	12·087	10·838	9·800	8·928
4–39	12·362	11·097	10·043	9·157
5–40	12·405	11·150	10·102	9·219
6–41	12·446	11·203	10·163	9·283
7–42	12·412	11·190	10·165	9·296
8–43	12·325	11·130	10·124	9·270
9–44	12·174	11·012	10·031	9·197
10–45	11·976	10·851	9·900	9·088
11–46	11·756	10·697	9·774	8·962
12–47	11·525	10·481	9·592	8·827
13–48	11·288	10·284	9·425	8·686
14–49	11·045	10·080	9·252	8·538
15–50	10·799	9·872	9·076	8·386
16–51	10·554	9·665	8·899	8·234
17–52	10·313	9·461	8·724	8·083
18–53	10·076	9·260	8·552	7·934
19–54	9·845	9·063	8·383	7·788
20–55	9·617	8·869	8·216	7·643
21–56	9·394	8·679	8·053	7·502
22–57	9·174	8·491	7·891	7·362
23–58	8·951	8·209	7·725	7·218
24–59	6·725	8·104	7·556	7·070
25–60	8·495	7·906	7·383	6·919
26–61	8·263	7·704	7·207	6·764
27–62	8·028	7·499	7·027	6·605
28–63	7·785	7·286	6·839	6·439
29–64	7·539	7·069	6·648	6·268
30–65	7·286	6·844	6·447	6·089

TABLE XXXV Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
31–66	7·028	6·615	6·243	5·905
32–67	6·768	6·382	6·033	5·717
33–68	6·504	6·146	5·820	5·524
34–69	6·259	5·906	5·603	5·326
35–70	5·971	5·663	5·382	5·125
36–71	5·703	5·419	5·159	4·920
37–72	5·435	5·174	4·934	4·714
38–73	5·169	4·930	4·710	4·507
39–74	4·908	4·690	4·488	4·301
40–75	4·656	4·457	4·272	4·101
41–76	4·420	4·238	4·069	3·912
42–77	4·184	4·019	3·865	3·722
43–78	3·942	3·794	3·655	3·525
44–79	3·685	3·552	3·428	3·312
45–80	3·426	3·308	3·197	3·093
46–81	3·176	3·072	2·973	2·881
47–82	2·936	2·843	2·756	2·673
48–83	2·714	2·632	2·554	2·481
49–84	2·544	2·470	2·400	2·334
50–85	2·388	2·322	2·258	2·198
51–86	2·248	2·188	2·131	2·077
52–87	2·117	2·063	2·012	1·963
53–88	2·008	1·960	1·914	1·870
54–89	1·858	1·817	1·778	1·740
55–90	1·666	1·633	1·601	1·570
56–91	1·402	1·377	1·353	1·330
57–92	1·120	1·102	1·085	1·069
58–93	·794	·784	·773	·763
59–94	·511	·505	·499	·494
60–95	·233	·230	·228	·226
61–96	·000	·000	·000	·000

TABLE XXXVI.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton.*

Difference of age 40 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1–41	9·523	8·585	7·800	7·135
2–42	10·907	9·839	8·942	8·182
3–43	11·343	10·242	9·315	8·528
4–44	11·578	10·468	9·531	8·733
5–45	11·597	10·500	9·571	8·778
6–46	11·610	10·528	9·609	8·823
7–47	11·550	10·491	9·589	8·815
8–48	11·435	10·404	9·524	8·767
9–49	11·260	10·263	9·409	8·673
10–50	11·014	10·085	9·260	8·548
11–51	10·816	9·894	9·100	8·411
12–52	10·582	9·698	8·934	8·270
13–53	10·344	9·497	8·763	8·123
14–54	10·100	9·290	8·586	7·970
15–55	9·851	9·077	8·403	7·812
16–56	9·595	8·858	8·214	7·648
17–57	9·340	8·639	8·024	7·481
18–58	9·089	8·422	7·835	7·316
19–59	8 841	8·207	7·648	7·153
20–60	8·597	7·995	7·463	6·990
21–61	8·357	7·787	7·281	6·830
22–62	8·119	7·580	7·100	6·670
23–63	7·874	7·365	6·910	6·503
24–64	7·626	7·147	6·717	6·331
25–65	7 370	6·920	6·515	6·151
26–66	7·110	6·689	6·309	5·966
27–67	6·847	6·454	6·098	5·776
28–68	6·581	6·215	5·883	5·581
29–69	6·313	5·973	5·664	5·383
30–70	6·043	5·729	5·442	5·180

TABLE XXXVI Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
31–71	5·772	5·483	5·218	4·974
32–72	5·502	5·236	4·992	4·707
33–73	5·235	4·991	4·706	4·559
34–74	4·973	4·749	4·543	4·353
35–75	4·720	4·516	4·327	4·152
36–76	4·481	4·295	4·123	3·962
37–77	4·242	4·073	3·910	3·770
38–78	3·996	3·844	3·702	3·570
39–79	3·734	3·598	3·471	3·352
40–80	3·469	3·349	3·236	3·130
41–81	3·216	3·109	3·009	2·914
42–82	2·973	2·878	2·789	2·705
43–83	2·750	2·066	2·587	2·511
44–84	2·581	2·505	2·433	2·365
45–85	2·424	2·356	2·291	2·230
46–86	2·282	2·221	2·162	2·107
47–87	2·148	2·093	2·041	1·991
48–88	2·036	1·987	1·941	1·895
49–89	1·882	1·840	1·800	1·761
50–90	1·685	1·651	1·619	1·590
51–91	1·417	1·391	1·367	1·343
52–92	1·130	1·113	1·005	1·079
53–93	·801	·790	·780	·779
54–94	·515	·509	·503	·498
55–95	·234	·232	·230	·228
56–96	·000	·000	·000	·000

TABLE XXXVII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton.*

Difference of age 45 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1–46	8·888	8·071	7·379	6·787
2–47	10·147	9·221	8·435	7·760
3–48	10·515	9·566	8·759	8·063
4–49	10·697	9·744	8·952	8·230
5–50	10·679	9·742	8·941	8·248
6–51	10·661	9·745	8·956	8·271
7–52	10·586	9·690	8·919	8·248
8–53	10·458	9·591	8·841	8·188
9–54	10·276	9·442	8·718	8·085
10–55	10·055	9·256	8·560	7·951
11–56	9·814	9·052	8·386	7·801
12–57	9·566	8·839	8·203	7·643
13–58	9·312	8·622	8·015	7·479
14–59	9·053	8·399	7·821	7·310
15–60	8·790	8·170	7·622	7·135
16–61	8·521	7·935	7·416	6·953
17–62	8·252	7·700	7·208	6·770
18–63	7·981	7·462	6·998	6·583
19–64	7·714	7·226	6·789	6·396
20–65	7·444	6·986	6·576	6·205
21–66	7·177	6·749	6·364	6·015
22–67	6·911	6·512	6·151	5·824
23–68	6·643	6·271	5·934	5·628
24–69	6·372	6·027	5·713	5·427
25–70	6·099	5·780	5·489	5·223
26–71	5·826	5·532	5·263	5·016
27–72	5·554	5·283	5·035	4·807
28–73	5·284	5·036	4·808	4·597
29–74	5·019	4·792	4·583	4·390
30–75	4·764	4·557	4·365	4·188

TABLE XXXVII Continued.

Ages	3 per Cent.	4 per Cent	5 pet Cent	6 per Cent
31–76	4·523	4·335	4·160	3·997
32–77	4·282	4·111	3·952	3·804
33–78	4·035	3·881	3·737	3·602
34–79	3·771	3·633	3·505	3·384
35–80	3·506	3·383	3·268	3·160
36–81	3·251	3·142	3·040	2·944
37–82	3·005	2·909	2·818	2·733
38–83	2·779	2·694	2·613	2 537
39–84	2·607	2·530	2·457	2·388
40–85	2·448	2·379	2·313	2·251
41–86	2·304	2·241	2·182	2·126
42–87	2·168	2·113	2·060	2·009
43–88	2·055	2·006	1·959	1·914
44–89	1·901	1·859	1·818	1·779
45–90	1·702	1·668	1·635	1·604
46–91	1·431	1·405	1·380	1·356
47–92	1·140	1·122	1·105	1·089
48–93	·808	·797	·786	·776
49–94	·519	·512	·507	·501
50–95	·235	·233	·231	·229
51–96	·000	·000	·000	·000

TABLE XXXVIII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton.*

Difference of age 50 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1–51	8·171	7·479	6·885	6·370
2–52	9·300	8·520	7·848	7·264
3–53	9·611	8·815	8·128	7·529
4–54	9·751	8·957	8·269	7·668
5–55	9·707	8·931	8·256	7·665
6–56	9·659	8·902	8·241	7·662
7–57	9·549	8·817	8·176	7·612
8–58	9·395	8·691	8·073	7·527
9–59	9·191	8·519	7·927	7·403
10–60	8·952	8·314	7·750	7·250
11–61	8·696	8·092	7·557	7·081
12–62	8·433	7·863	7·357	6·905
13–63	8·161	7·625	7·147	6·719
14–64	7·884	7·381	6·931	6·527
15–65	7·597	7·127	6·705	6·325
16–66	7·304	6·866	6·472	6·115
17–67	7·012	6·604	6·236	5·903
18–68	6·721	6·343	6·001	5·689
19–69	6·434	6·084	5·766	5·476
20–70	6·149	5·826	5·532	5·262
21–71	5·870	5·572	5·300	5·050
22–72	5·595	5·321	5·070	4·840
23–73	5·323	5·072	4·841	4·628
24–74	5·056	4·827	4·615	4·419
25–75	4·799	4·589	4·396	4·216
26–76	4·556	4·365	4·188	4·024
27–77	4·313	4·140	3·979	3·829
28–78	4·064	3·908	3·762	3·626
29–79	3·798	3·659	3·528	3·406
30–80	3·530	3·406	3·290	3·181

TABLE XXXVIII Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
31–81	3·274	3·164	3·060	2·963
32–82	3·027	2·929	2·838	2·751
33–83	2·800	2·713	2·632	2·555
34–84	2·627	2·549	2·476	2·406
35–85	2·468	2·398	2·331	2·268
36–86	2·323	2·260	2·200	2·143
37–87	2·187	2·130	2·077	2·026
38–88	2·072	2·022	1·974	1·929
39–89	1·915	1·872	1·832	1·792
40–90	1·713	1·679	1·646	1·614
41–91	1·439	1·413	1·388	1·364
42–92	1·146	1·128	1·111	1·094
43–93	·811	·800	·790	·779
44–94	·521	·515	·509	·503
45–95	·236	·234	·232	·230
46–96	·000	·000	·000	·000

TABLE XXXIX.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton.*

Difference of age 55 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1–56	7·412	6·843	6·346	5·911
2–57	8·392	7·756	7·199	6·709
3–58	8·630	7·986	7·421	6·922
4–59	8·712	8·075	7·514	7·017
5–60	8·629	8·011	7·466	6·982
6–61	8·542	7·944	7·415	6·945
7–62	8·400	7·828	7·319	6·865
8–63	8·214	7·669	7·184	6·750
9–64	7·984	7·470	7·010	6·598
10–65	7·718	7·236	6·803	6·414
11–66	7·437	6·987	6·581	6·215
12–67	7·149	6·730	6·351	6·009
13–68	6·857	6·468	6·116	5·796
14–69	6·562	6·202	5·876	5·578
15–70	6·264	5·933	5·631	5·355
16–71	5·964	5·660	5·382	5·127
17–72	5·667	5·389	5·133	4·899
18–73	5·378	5·123	4·889	4·673
19–74	5·098	4·866	4·651	4·453
20–75	4·831	4·619	4·424	4·242
21–76	4·583	4·391	4·212	4·046
22–77	4·339	4·164	4·001	3·850
23–78	4·087	3·930	3·783	3·646
24–79	3·820	3·679	3·548	3·424
25–80	3·550	3·425	3·308	3·198
26–81	3·202	3·181	3·077	2·979
27–82	3·043	2·945	2·853	2·765
28–83	2·815	2·728	2·646	2·568
29–84	2·641	2·563	2·489	2·418
30–85	2·481	2·411	2·344	2·280

TABLE XXXIX Continued.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
31–86	2·336	2·272	2·212	2·154
32–87	2·198	2·142	2·088	2·036
33–88	2·083	2·033	1·985	1·939
34–89	1·925	1·882	1·841	1·802
35–90	1·723	1·688	1·654	1·622
36–91	1·446	1·420	1·395	1·371
37–92	1·152	1·134	1·116	1·099
38–93	·815	·804	·793	·783
39–94	·523	·517	·511	·505
40–95	·237	·235	·233	·231
41–96	·000	·000	·000	·000

TABLE XL.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton*.

Difference of age 60 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1–61	6·571	6·123	5·725	5·372
2–62	7·391	6·894	6·452	6·059
3–63	7·545	7·048	6·605	6·209
4–64	7·562	7·076	6·641	6·251
5–65	7·429	6·963	6·546	6·171
6–66	7·290	6·846	6·447	6·087
7–67	7·104	6·684	6·306	5·963
8–68	6·884	6·490	6·134	5·811
9–69	6·628	6·262	5·929	5·626
10–70	6·347	6·008	5·700	5·418
11–71	6·056	5·744	5·460	5·199
12–72	5·763	5·478	5·216	4·976
13–73	5·473	5·212	4·972	4·751
14–74	5·188	4·950	4·731	4·528
15–75	4·911	4·695	4·495	4·310
16–76	4·649	4·452	4·270	4·101
17–77	4·388	4·210	4·045	3·892
18–78	4·123	3·964	3·815	3·677
19–79	3·846	3·704	3·571	3·447
20–80	3·569	3·443	3·325	3·214
21–81	3·307	3·195	3·091	2·992
22–82	3·057	2·958	2·865	2·777
23–83	2·828	2·740	2·657	2·579
24–84	2·653	2·574	2·499	2·429
25–85	2·492	2·421	2·354	2·290
26–86	2·346	2·282	2·221	2·163
27–87	2·208	2·151	2·096	2·044
28–88	2·091	2·041	1·992	1·946
29–89	1·933	1·889	1·848	1·808
30–90	1·729	1·694	1·660	1·628
31–91	1·451	1·425	1·400	1·376
32–92	1·155	1·137	1·119	1·102
33–93	·817	·806	·795	·785
34–94	·524	·518	·512	·506
35–95	·238	·235	·233	·231

TABLE XLI.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton.*

Difference of age 65 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1–66	5·633	5·265	4·996	4·728
2–67	6·266	5·895	5·569	5·270
3–68	6·330	5·965	5·641	5·352
4–69	6·277	5·924	5·611	5·332
5–70	6·102	5·768	5·472	5·209
6–71	5·925	5·610	5·331	5·084
7–72	5·714	5·418	5·157	4·929
8–73	5·480	5·204	4·963	4·752
9–74	5·225	4·969	4·747	4·556
10–75	4·962	4·725	4·522	4·350
11–76	4·707	4·487	4·301	4·148
12–77	4·449	4·368	4·195	3·943
13–78	4·185	4·022	3·871	3·729
14–79	3·904	3·759	3·624	3·497
15–80	3·621	3·492	3·372	3·259
16–81	3·348	3·235	3·128	3·028
17–82	3·087	2·987	2·893	2·804
18–83	2·849	2·760	2·677	2·598
19–84	2·668	2·589	2·513	2·442
20–85	2·503	2·431	2·364	2·299
21–86	2·354	2·290	2·229	2·171
22–87	2·216	2·158	2·104	2·051
23–88	2·099	2·048	1·999	1·953
24–89	1·939	1·895	1·854	1·814
25–90	1·734	1·699	1·665	1·633
26–91	1·455	1·429	1·404	1·379
27–92	1·158	1·140	1·122	1·105
28–93	·819	·808	·797	·786
29–94	·525	·519	·513	·507
30–95	·238	·236	·234	·231
31–96	·000	·000	·000	·000

TABLE XLII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made at *Northampton.*

Difference of age 70 years.

Ages	3 per Cent	4 per Cent	5 per Cent	6 per Cent
1–71	4·611	4·380	4·169	3·976
2–72	5·061	4·814	4·588	4·380
3–73	5·051	4·811	4·591	4·389
4–74	4·953	4·726	4·516	4·323
5–75	4·768	4·557	4·362	4·181
6–76	4·599	4·403	4·221	4·053
7–77	4·402	4·222	4·055	3·899
8–78	4·180	4·016	3·864	3·722
9–79	3·921	3·775	3·638	3·510
10–80	3·647	3·517	3·395	3·281
11–81	3·380	3·264	3·156	3·054
12–82	3·122	3·020	2·924	2·833
13–83	2·884	2·794	2·709	2·628
14–84	2·703	2·622	2·545	2·472
15–85	2·535	2·462	2·393	2·327
16–86	2·380	2·315	2·253	2·194
17–87	2·235	2·177	2·121	2·069
18–88	2·112	2·061	2·012	1·965
19–89	1·946	1·904	1·862	1·822
20–90	1·739	1·704	1·670	1·638
21–91	1·459	1·432	1·407	1·382
22–92	1·160	1·142	1·124	1·107
23–93	·820	·809	·798	·788
24–94	·526	·520	·514	·508
25–95	·238	·236	·234	·232
26–96	·000	·000	·000	·000

TABLE XLIII.

Showing the value of an annuity on *Three equal* Joint Lives, deduced from the observations made at *Northampton.*

Common Age	4 per Cent	Common Age	4 per Cent	Common Age	4 per Cent
1	5·309	36	8·448	71	2·810
2	8·251	37	8·309	72	2·627
3	9·632	38	8·165	73	2·448
4	10·661	39	8·017	74	2·277
5	11·170	40	7·865	75	2·119
6	11·707	41	7·714	76	1·985
7	12·058	42	7·567	77	1·855
8	12·266	43	7·423	78	1·720
9	12·298	44	7·275	79	1·563
10	12·200	45	7·126	80	1·400
11	12·043	46	6·972	81	1·245
12	11·865	47	6·813	82	1·092
13	11·678	48	6·650	83	·949
14	11·481	49	6·482	84	·860
15	11·274	50	6·317	85	·782
16	11·056	51	6·161	86	·716
17	10·845	52	6·011	87	·662
18	10·656	53	5·859	88	·646
19	10·490	54	5·705	89	·614
20	10·342	55	5·550	90	·563
21	10·222	56	5·393	91	·452
22	10·118	57	5·235	92	·337
23	10·012	58	5·076	93	·185
24	9·905	59	4·916	94	·085
25	9·796	60	4·755	95	·015
26	9·685	61	4·593		
27	9·572	62	4·432		
28	9·457	63	4·263		
29	9·340	64	4·095		
30	9·221	65	3·914		
31	9·099	66	3·733		
32	8·975	67	3·550		
33	8·848	68	3·366		
34	8·718	69	3·181		
35	8·585	70	2·995		

TABLE XLIV.

Showing the value of an annuity on *Three* Joint Lives, whose differences of age are 10 and 20 years, deduced from the observations made at *Northampton.*

Ages	4 per Cent	Ages	4 per Cent
1–11–21	8·627	41–51–61	5·827
2–12–22	9·914	42–52–62	5·662
3–13–23	10·344	43–53–63	5·494
4–14–24	10·598	44–54–64	5·322
5–15–25	10·655	45–55–65	5·145
6–16–26	10·708	46–56–66	4·965
7–17–27	10·700	47–57–67	4·782
8–18–28	10·654	48–58–68	4·597
9–19–29	10·562	49–59–69	4·408
10–20–30	10·438	50–60–70	4·219
11–21–31	10·305	51–61–71	4·032
12–22–32	10·170	52–62–72	3·847
13–23–33	10·031	53–63–73	3·660
14–24–34	9·887	54–64–74	3·477
15–25–35	9·738	55–65–75	3·298
16–26–36	9·584	56–66–76	3·128
17–27–37	9·429	57–67–77	2·959
18–28–38	9·278	58–68–78	2·785
19–29–39	9·131	59–69–79	2·598
20–30–40	8·986	60–70–80	2·408
21–31–41	8·850	61–71–81	2·224
22–32–42	8·718	62–72–82	2·044
23–33–43	8·586	63–73–83	1·875
24–34–44	8·451	64–74–84	1·743
25–35–45	8·313	65–75–85	1·623
26–36–46	8·171	66–76–86	1·519
27–37–47	8·027	67–77–87	1·425
28–38–48	7·878	68–78–88	1·350
29–39–49	7·725	69–79–89	1·248
30–40–50	7·571	70–80–90	1·122
31–41–51	7·420	71–81–91	·951
32–42–52	7·272	72–82–92	·767
33–43–53	7·123	73–83–93	·548
34–44–54	6·971	74–84–94	·362
35–45–55	6·816	75–85–95	·169
36–46–56	6·658		
37–47–57	6·497		
38–48–58	6·332		
39–49–59	6·164		
40–50–60	5·994		

TABLE XLV.

Showing the number of persons *living* and *dying* at every age, according to the observations made in *London*.

Age	Living	Dying	Age	Living	Dying	Age	Living	Dying
0	1000	320						
1	680	133	31	294	7	61	97	5
2	547	51	32	287	7	62	92	5
3	496	27	33	280	7	63	87	5
4	469	17	34	273	7	64	82	5
5	452	12	35	266	7	65	77	5
6	440	10	36	259	7	66	72	5
7	430	8	37	252	7	67	67	5
8	422	7	38	245	8	68	62	4
9	415	5	39	237	8	69	58	4
10	410	5	40	229	7	70	54	4
11	405	5	41	222	8	71	50	4
12	400	5	42	214	8	72	46	4
13	395	5	43	206	7	73	42	3
14	390	5	44	199	7	74	39	3
15	385	5	45	192	7	75	36	3
16	380	5	46	185	7	76	33	3
17	375	5	47	178	7	77	30	3
18	370	5	48	171	6	78	27	2
19	365	5	49	165	6	79	25	
20	360	5	50	159	6			
21	355	5	51	153	6			
22	350	5	52	147	6			
23	345	6	53	141	6			
24	339	6	54	135	6			
25	333	6	55	129	6			
26	327	6	56	123	6			
27	321	6	57	117	5			
28	315	7	58	112	5			
29	308	7	59	107	5			
30	301	7	60	102	5			

TABLE XLVI.

Showing the *Expectation of Life*, deduced from the observations made in *London.*

Age	Expectation	Age	Expectation	Age	Expectation
0	19·2				
1	27·0	31	23·1	61	12·0
2	32·0	32	22·7	62	11·6
3	34·0	33	22·3	63	11·2
4	35·6	34	21·9	64	10·8
5	36·0	35	21·5	65	10·5
6	36·0	36	21·1	66	10·1
7	35·8	37	20·7	67	9·8
8	35·6	38	20·3	68	9·4
9	35·2	39	19·9	69	9·1
10	34·8	40	19·6	70	8·8
11	34·3	41	19·2	71	8·4
12	33·7	42	18·8	72	8·1
13	33·1	43	18·5	73	7·8
14	32·5	44	18·1	74	7·5
15	31·9	45	17·8	75	7·2
16	31·3	46	17·4	76	6·8
17	30·7	47	17·0	77	6·4
18	30·1	48	16·7	78	6·0
19	29·5	49	16·3	79	5·5
20	28·9	50	16·0	80	5·0
21	28·3	51	15·6		
22	27·7	52	15·2		
23	27·2	53	14·9		
34	26·6	54	14·5		
25	26·1	55	14·2		
26	25·6	56	13·8		
27	25·1	57	13·4		
28	24·6	58	13·1		
29	24·1	59	12·7		
30	23·6	60	12·4		

TABLE XLVII.

Showing the value of an annuity on a Single Life, at every age, deduced from the observations made in *London*.

Age	3 per Cent	4 per Cent	5 per Cent	Age	3 per Cent	4 per Cent	5 per Cent	Age	3 per Cent	4 per Cent	5 per Cent
6	18·8	16·2	14·1	31	14·8	12·9	11·4	56	10·1	9·1	8·4
7	18·9	16·3	14·2	32	14·6	12·7	11·3	57	9·9	8·9	8·2
8	19·0	16·4	14·3	33	14·4	12·6	11·2	58	9·6	8·7	8·1
9	19·0	16·4	14·3	34	14·2	12·4	11·0	59	9·4	8·6	8·0
10	19·0	16·4	14·3	35	14·1	12·3	10·9	60	9·2	8·4	7·9
11	19·0	16·4	14·3	36	13·9	12·1	10·8	61	8·9	8·2	7·7
12	18·9	16·3	14·2	37	13·7	11·9	10·6	62	8·7	8·1	7·6
13	18·7	16·2	14·1	38	13·5	11·8	10·5	63	8·5	7·9	7·4
14	18·5	16·0	14·0	39	13·3	11·6	10·4	64	8·3	7·7	7·3
15	18·3	15·8	13·9	40	13·2	11·5	10·3	65	8·0	7·5	7·1
16	18·1	15·6	13·7	41	13·0	11·4	10·2	66	7·8	7·3	6·9
17	17·9	15·4	13·5	42	12·8	11·2	10·1	67	7·6	7·1	6·7
18	17·6	15·2	13·4	43	12·6	11·1	10·0	68	7·4	6·9	6·6
19	17·4	15·0	13·2	44	12·5	11·0	9·9	69	7·1	6·7	6·4
20	17·2	14·8	13·0	45	12·3	10·8	9·8	70	6·9	6·5	6·2
21	17·0	14·7	12·9	46	12·1	10·7	9·7	71	6·7	6·3	6·0
22	16·8	14·5	12·7	47	11·9	10·5	9·5	72	6·5	6·1	5·8
23	16·5	14·3	12·6	48	11·8	10·4	9·4	73	6·2	5·9	5·6
24	16·3	14·1	12·4	49	11·6	10·2	9·3	74	5·9	5·6	5·4
25	16·1	14·0	12·3	50	11·4	10·1	9·2	75	5·6	5·4	5·2
26	15·9	13·8	12·1	51	11·2	9·9	9·0				
27	15·6	13·6	12·0	52	11·0	9·8	8·9				
28	15·4	13·4	11·8	53	10·7	9·6	8·8				
29	15·2	13·2	11·7	54	10·5	9·4	8·6				
30	15·0	13·1	11·6	55	10·3	9·3	8·5				

TABLE XLVIII.

Showing the value of an annuity on Two Joint Lives, deduced from the observations made in *London*.

Ages		3 per Cent	4 per Cent	5 per Cent
10	10	14·7	13·0	11·6
	15	14·3	12·7	11·3
	20	13·8	12·2	10·8
	25	13·1	11·6	10·2
	30	12·3	10·9	9·7
	35	11·5	10·2	9·1
	40	10·7	9·6	8·6
	45	10·0	9·0	8·1
	50	9·3	8·4	7·6
	55	8·6	7·8	7·1
	60	7·8	7·2	6·6
	65	6·9	6·5	6·1
	70	6·1	5·8	5·5
	75	5·3	5·1	4·9
15	15	13·0	12·3	11·0
	20	13·3	11·8	10·5
	25	12·6	11·2	10·1
	30	11·9	10·6	9·5
	35	11·2	10·0	9·0
	40	10·4	9·4	8·5
	45	9·6	8·8	8·0
	50	8·9	8·2	7·5
	55	8·2	7·6	7·0
	60	7·5	7·0	6·5
	65	6·8	6·4	6·0
	70	6·0	5·7	5·4
	75	5·2	5·0	4·8
20	20	12·8	11·3	10·1
	25	12·2	10·8	9·7
	30	11·6	10·3	9·2
	35	10·9	9·8	8·8
	40	10·2	9·2	8·4
	45	9·5	8·6	7·9
	50	8·8	8·0	7·4
	55	8·1	7·5	6·9
	60	7·4	6·9	6·4
	65	6·7	6·3	5·9
	70	6·0	5·7	5·4
	75	5·2	5·0	4·8
25	25	11·8	10·5	9·4
	30	11·3	10·1	9·0
	35	10·7	9·6	8·6
	40	10·0	9·1	8·2
	45	9·4	8·5	7·8
	50	8·7	7·9	7·3
	55	8·0	7·4	6·8
	60	7·3	6·8	6·3
	65	6·6	6·2	5·8
	70	5·9	5·6	5·3
	75	5·1	4·9	4·7
30	30	10·8	9·6	8·6
	35	10·3	9·2	8·3
	40	9·7	8·8	8·0
	45	9·1	8·3	7·6
	50	8·5	7·8	7·2
	55	7·9	7·3	6·7
	60	7·2	6·7	6·2
	65	6·5	6·1	5·7
	70	5·8	5·5	5·2
	75	5·1	4·9	4·7

TABLE XLVIII Continued.

Ages		3 per Cent	4 per Cent	5 per Cent
35	35	9·9	8·8	8·0
	40	9·4	8·5	7·7
	45	8·9	8·1	7·4
	50	8·3	7·6	7·0
	55	7·7	7·1	6·6
	60	7·1	6·5	6·1
	65	6·4	6·0	5·6
	70	5·7	5·4	5·1
	75	5·0	4·8	4·6
40	40	9·1	8·1	7·3
	45	8·7	7·8	7·1
	50	8·2	7·4	6·8
	55	7·6	6·9	6·4
	60	7·0	6·4	6·0
	65	6·4	5·9	5·5
	70	5·7	5·4	5·1
	75	5·0	4·8	4·6
45	45	8·3	7·4	6·7
	50	7·9	7·1	6·5
	55	7·4	6·7	6·2
	60	6·8	6·3	5·8
	65	6·3	5·8	5·4
	70	5·6	5·3	5·0
	75	4·9	4·7	4·5
50	50	7·6	6·8	6·2
	55	7·2	6·5	6·0
	60	6·7	6·1	5·7
	65	6·2	5·7	5·3
	70	5·5	5·2	4·9
	75	4·8	4·6	4·4
55	55	6·9	6·2	5·7
	60	6·5	5·9	5·5
	65	6·0	5·6	5·2
	70	5·4	5·1	4·8
	75	4·7	4·5	4·3
60	60	6·1	5·6	5·2
	65	5·7	5·3	4·9
	70	5·2	4·9	4·6
	75	4·6	4·4	4·2
65	65	5·4	5·0	4·7
	70	4·9	4·6	4·4
	75	4·4	4·2	4·0
70	70	4·6	4·4	4·2
	75	4·2	4·0	3·9
75	75	3·8	3·7	3·6

TABLE XLIX.

Showing the value of an annuity on a Single Life, according to the hypothesis of *M. De Moivre.*

Age	3 per Cent	$3\frac{1}{2}$ per Cent	4 per Cent	$4\frac{1}{2}$ per Cent	5 per Cent	6 per Cent
8	19·736	18·160	16·791	15·595	14·544	12·790
9	19·868	18·269	16·882	15·672	14·607	12·839
10	19·868	18·269	16·882	15·672	14·607	12·839
11	19·736	18·160	16·791	15·595	14·544	12·790
12	19·604	18·049	16·698	15·517	14·480	12·741
13	19·469	17·937	16·604	15·437	14·412	12·691
14	19·331	17·823	16·508	15·356	14·342	12·639
15	19·192	17·707	16·410	15·273	14·271	12·586
16	19·050	17·588	16·311	15·189	14·197	12·532
17	18·905	17·467	16·209	15·102	14·123	12·476
18	18·759	17·344	16·105	15·015	14·047	12·419
19	18·610	17·220	15·999	14·923	13·970	12·361
20	18·458	17·093	15·891	14·831	13·891	12·301
21	18·305	16·963	15·781	14·737	13·810	12·239
22	18·148	16·830	15·669	14·641	13·727	12·177
23	17·990	16·696	15·554	14·543	13·642	12·112
24	17·827	16·559	15·437	14·442	13·555	12·045
25	17·664	16·419	15·318	14·340	13·466	11·978
26	17·497	16·277	15·197	14·235	13·375	11·908
27	17·327	16·133	15·073	14·128	13·282	11·837
28	17·154	15·985	14·946	14·018	13·186	11·763
29	16·979	15·835	14·816	13·905	13·088	11·688
30	16·800	15·682	14·684	13·791	12·988	11·610
31	16·620	15·526	14·549	13·673	12·885	11·530
32	16·436	15·367	14·411	13·553	12·780	11·449
33	16·248	15·204	14·270	13·430	12·673	11·365
34	16·057	15·039	14·126	13·304	12·562	11·278
35	15·864	14·871	13·979	13·175	12·449	11·189
36	15·666	14·699	13·829	13·044	12·333	11·098
37	15·465	14·524	13·676	12·909	12·214	11·003
38	15·260	14·345	13·519	12·771	12·091	10·907
39	15·053	14·163	13·359	12·630	11·966	10·807
40	14·842	13·978	13·196	12·485	11·837	10·704
41	14·626	13·789	13·028	12·337	11·705	10·599
42	14·407	13·596	12·858	12·185	11·570	10·490
43	14·185	13·399	12·683	12·029	11·431	10·378
44	13·958	13·199	12·504	11·870	11·288	10·263
45	13·728	12·993	12·322	11·707	11·142	10·144

TABLE XLIX Continued.

Age	3 per Cent	3½ per Cent	4 per Cent	4½ per Cent	5 per Cent	6 per Cent
46	13·493	12·784	12·135	11·540	10·992	10·021
47	13·254	12·571	11·944	11·368	10·837	9·895
48	13·012	12·354	11·748	11·192	10·679	9·705
49	12·764	12·131	11·548	11·012	10·515	9·630
50	12·511	11·904	11·344	10·827	10·348	9·492
51	12·255	11·673	11·135	10·638	10·176	9·349
52	11·994	11·437	10·921	10·443	9·999	9·201
53	11·729	11·195	10·702	10·243	9·817	9·049
54	11·457	10·950	10·478	10·039	9·630	8·891
55	11·183	10·698	10·248	9·829	9·437	8·729
56	10·902	10·443	10·014	9·614	9·239	8·561
57	10·616	10·181	9·773	9·393	9·036	8·387
58	10·325	9·913	9·527	9·166	8·826	8·208
59	10·029	9·640	9·275	8·933	8·611	8·023
60	9·727	9·361	9·017	8·694	8·389	7·831
61	9·419	9·076	8·753	8·449	8·161	7·633
62	9·107	8·786	8·482	8·197	7·926	7·428
63	8·787	8·488	8·205	7·938	7·684	7·216
64	8·462	8·185	7·921	7·672	7·435	6·997
65	8·132	7·875	7·631	7·399	7·179	6·770
66	7·794	7·558	7·333	7·119	6·915	6·535
67	7·450	7·234	7·027	6·831	6·643	6·292
68	7·099	6·902	6·714	6·534	6·362	6·040
69	6·743	6·565	6·394	6·230	6·073	5·779
70	6·378	6·219	6·065	5·918	5·775	5·508
71	6·008	5·865	5·728	5·596	5·468	5·228
72	5·631	5·505	5·383	5·265	5·152	4·937
73	5·246	5·136	5·029	4·926	4·826	4·636
74	4·854	4·759	4·666	4·576	4·489	4·324
75	4·453	4·373	4·293	4·217	4·143	4·000
76	4·046	3·978	3·912	3·847	3·784	3·664
77	3·632	3·575	3·520	3·467	3·415	3·315
78	3·207	3·163	3·111	3·076	3·034	2·953
79	2·776	2·741	2·707	2·673	2·641	2·578
80	2·334	2·309	2·284	2·259	2·235	2·188
81	1·886	1·867	1·850	1·832	1·816	1·783
82	1·429	1·411	1·406	1·394	1·384	1·362
83	·961	·955	·950	·943	·937	·925
84	·484	·483	·481	·479	·476	·472
85	·000	·000	·000	·000	·000	·000

TABLE L.

Showing the value of an annuity on Two Joint Lives, according to the hypothesis of *M. De Moivre.*

Ages		3 per Cent	4 per Cent	5 per Cent
10	10	15·206	13·342	11·855
	15	14·878	13·093	11·661
	20	14·503	12·808	11·430
	25	14·074	12·480	11·182
	30	13·585	12·102	10·884
	35	13·025	11·665	10·537
	40	12·381	11·156	10·128
	45	11·644	10·564	9·646
	50	10·796	9·871	9·074
	55	9·822	9·059	8·391
	60	8·704	8·105	7·572
	65	7·417	6·980	6·585
	70	5·936	5·652	5·391
15	15	14·574	12·860	11·478
	20	14·225	12·593	11·266
	25	13·822	12·281	11·022
	30	13·359	11·921	10·736
	35	12·824	11·501	10·402
	40	12·207	11·013	10·008
	45	11·496	10·440	9·541
	50	10·675	9·767	8·985
	55	9·727	8·975	8·318
	60	8·632	8·041	7·515
	65	7·377	6·934	6·544
	70	5·932	5·623	5·364
20	20	13·904	12·341	11·067
	25	13·531	12·051	10·840
	30	13·098	11·711	10·565
	35	12·594	11·314	10·278
	40	12·008	10·847	9·870
	45	11·325	10·297	9·420
	50	10·536	9·648	8·880
	55	9·617	8·879	8·233

TABLE L Continued.

Ages		3 per Cent	4 per Cent	5 per Cent
20	60	8·549	7·967	7·448
	65	7·308	6·882	6·495
	70	5·868	5·590	5·333
25	25	13·192	11·786	10·621
	30	12·794	11·468	10·367
	35	12·333	11·093	10·067
	40	11·776	10·655	9·708
	45	11·130	10·131	9·278
	50	10·374	9·509	8·761
	55	9·488	8·766	8·134
	60	8·452	7·880	7·371
	65	7·241	6·826	6·440
	70	5·826	5·551	5·294
30	30	12·434	11·182	10·133
	35	12·010	10·838	9·854
	40	11·502	10·428	9·514
	45	10·898	9·936	9·112
	50	10·183	9·345	8·620
	55	9·338	8·634	8·018
	60	8·338	7·779	7·280
	65	7·161	6·748	6·373
	70	5·777	5·505	5·254
35	35	11·632	10·530	9·600
	40	11·175	10·157	9·291
	45	10·622	9·702	8·913
	50	9·955	9·149	8·450
	55	9·156	8·476	7·879
	60	8·202	7·658	7·172
	65	7·066	6·662	6·294
	70	5·718	5·450	5·203
40	40	10·777	9·826	9·014
	45	10·283	9·418	8·671
	50	9·677	8·911	8·244
	55	8·936	8·283	7·710
	60	8·038	7·510	7·039
	65	6·951	6·556	6·198
	70	5·646	5·383	5·141

TABLE L Continued.

Ages		3 per Cent	4 per Cent	5 per Cent
45	45	9·863	9·063	8·370
	50	9·331	8·619	7·987
	55	8·662	8·044	7·500
	60	7·831	7·332	6·875
	65	6·807	6·425	6·080
	70	5·556	5·300	5·063
50	50	8·892	8·235	7·660
	55	8·312	7·738	7·230
	60	7·568	7·091	6·664
	65	6·623	6·258	5·926
	70	5·442	5·193	4·964
55	55	7·849	7·332	6·873
	60	7·220	6·781	6·386
	65	6·379	6·036	5·724
	70	5·201	5·053	4·833
60	60	6·737	6·351	6·001
	65	6·043	5·730	5·444
	70	5·081	4·858	4·653
65	65	5·547	5·277	5·031
	70	4·773	4·571	4·385
70	70	4·270	4·104	3·952

TABLE LI.

Showing the value of an Assurance of £100 on a Single Life, (either in Single or Annual Payments) deduced from the observations made at *Northampton*, and at 3 per cent interest.

This Table is adopted by *all* the Assurance Offices in London.

Age	Single Premium	Annual Premium	Age	Single Premium	Annual Premium
8 to 14	....	1·879	41	54·505	3·487
15	39·834	1·929	42	55·172	3·583
16	40·481	1·983	43	55·839	3·683
17	41·113	2·033	44	56·517	3·787
18	41·710	2·083	45	57·208	3·896
19	42·272	2·133	46	57·913	4·008
20	42·802	2·179	47	58·632	4·129
21	43·291	2·225	48	59·366	4·254
22	43·756	2·267	49	60·117	4·392
23	44·229	2·312	50	60·866	4·533
24	44·710	2·354	51	61·603	4·675
25	45·202	2·403	52	62·340	4·821
26	45·703	2·450	53	63·086	4·979
27	46·213	2·504	54	63·784	5·142
28	46·732	2·554	55	64·612	5·317
29	47·261	2·612	56	65·392	5·504
30	47·800	2·671	57	66·182	5·700
31	48·353	2·725	58	66·980	5·908
32	48·913	2·787	59	67·792	6·133
33	49·486	2·854	60	68·611	6·367
34	50·072	2·921	61	69·438	6·617
35	50·666	2·992	62	70·277	6·887
36	51·275	3·067	63	71·136	7·179
37	51·898	3·142	64	72·007	7·492
38	52·530	3·225	65	72·901	7·837
39	53·180	3·308	66	73·804	8·204
40	53·841	3·396	67	74·713	8·604

TABLE LII.

Showing the value of an assurance of £100 on Two Joint Lives, (either in Single or Annual Payments) deduced from the observations made at *Northampton*, and at 3 per cent interest.

This Table is adopted by *all* the Assurance Offices in London.

Ages		Single Premium	Annual Premium
10	10	49·498	2·855
	15	51·177	3·053
	20	52·958	3·279
	25	54·319	3·463
	30	55·873	3·688
	35	57·693	3·972
	40	59·832	4·339
	45	62·206	4·794
	50	64·919	5·390
	55	67·801	6·133
	60	71·012	7·135
	65	74·606	8·557
15	15	52·731	3·249
	20	54·388	3·473
	25	55·641	3·653
	30	57·083	3·874
	35	58·783	4·154
	40	60·799	4·517
	45	63·047	4·969
	50	65·634	5·563
	55	68·395	6·303
	60	71·485	7·302
	65	74·960	8·719
20	20	55·923	3·695
	25	57·065	3·871
	30	58·390	4·087
	35	59·968	4·363
	40	61·856	4·723
	45	63·979	5·173

Ages		Single Premium	Annual Premium
20	50	66·438	5·766
	55	69·077	6·506
	60	72·049	7·508
	65	75·406	8·930
25	25	58·106	4·040
	30	59·322	4·248
	35	60·786	4·515
	40	62·559	4·867
	45	64·571	5·308
	50	66·923	5·893
	55	69·461	6·625
	60	72·343	7·619
	65	75·621	9·035
30	30	60·418	4·446
	35	61·754	4·703
	40	63·392	5·044
	45	65·271	5·474
	50	67·495	6·048
	55	69·915	6·769
	60	72·685	7·751
	65	75·866	9·156
35	35	62·944	4·947
	40	64·428	5·275
	45	66·149	5·692
	50	68·217	6·252
	55	70·492	6·958
	60	73·125	7·925
	65	76·181	9·316

TABLE LII Continued.

Ages		Single Premium	Annual Premium	Ages		Single Premium	Annual Premium
40	40	65·736	5·588	50	50	71·705	7·381
	45	67·274	5·988		55	73·344	8·014
	50	69·154	6·530		60	75·357	8·907
	55	71·250	7·218		65	77·831	10·226
	60	73·713	8·168	55	55	74·713	8·606
	65	76·612	9·541		60	76·443	9·451
45	45	68·611	6·367		65	78·637	10·721
	50	70·278	6·887	60	60	77·846	10·235
	55	72·164	7·551		65	79·699	11·434
	60	74·424	8·476	65	65	81·152	12·541
	65	77·134	9·825				

TABLE LIII.

Showing the value of an Assurance of £100 to be received on the decease of A, provided he dies *before* B; deduced (according to Mr. Simpson's rule) from the observations made at *Northampton*, and at 3 per cent. interest.

This Table is adopted by *all* the Assurance Offices in London; although it is one of the most *inaccurate* that ever was published. See page 511.

Age of A	Age of B	Single Premium	Annual Premium	Age of A	Age of B	Single Premium	Annual Premium
10	10	24·749	1·427	20	10	29·461	1·824
	15	24·198	1·444		15	28·786	1·838
	20	23·498	1·455		20	27·961	1·848
	25	22·531	1·437		25	26·811	1·819
	30	21·468	1·417		30	25·546	1·788
	35	20·317	1·399		35	24·176	1·759
	40	19·070	1·383		40	22·692	1·733
	45	17·696	1·364		45	21·058	1·703
	50	16·214	1·346		50	19·294	1·674
	55	14·631	1·324		55	17·410	1·640
	60	12·925	1·299		60	15·381	1·603
	65	11·098	1·273		65	13·206	1·564
	70	9·153	1·246		70	10·892	1·523
15	10	26·979	1·609	25	10	31·789	2·027
	15	26·365	1·625		15	31·093	2·042
	20	25·602	1·635		20	30·254	2·052
	25	24·549	1·612		25	29·053	2·020
	30	23·391	1·588		30	27·683	1·982
	35	22·136	1·564		35	26·198	1·946
	40	20·778	1·544		40	24·590	1·913
	45	19·281	1·520		45	22·819	1·876
	50	17·666	1·497		50	20·907	1·841
	55	15·941	1·469		55	18·866	1·799
	60	14·083	1·439		60	16·667	1·755
	65	12·092	1·407		65	14·310	1·710
	70	9·973	1·373		70	11·803	1·662

TABLE LIII Continued.

Age of A	Age of B	Single Premium	Annual Premium	Age of A	Age of B	Single Premium	Annual Premium
30	10	34·404	2·271	45	10	44·511	3·430
	15	33·694	2·287		15	43·766	3·450
	20	32·843	2·299		20	42·921	3·471
	25	31·640	2·266		25	41·753	3·433
	30	30·209	2·223		30	40·369	3·386
	35	28·589	2·177		35	38·735	3·333
	40	26·834	2·135		40	36·775	3·273
	45	24·901	2·088		45	34·306	3·183
	50	22·815	2·044		50	31·432	3·080
	55	20·588	1·993		55	28·364	2·968
	60	18·188	1·939		60	25·057	2·854
	65	15·616	1·885		65	21·514	2·740
	70	12·880	1·829		70	17·744	2·629
35	10	37·375	2·573	50	10	48·705	4·044
	15	36·647	2·590		15	47·968	4·066
	20	35·794	2·604		20	47·144	4·091
	25	34·588	2·569		25	46·017	4·052
	30	33·166	2·526		30	44·680	4·004
	35	31·472	2·474		35	43·101	3·950
	40	29·540	2·419		40	41·208	3·891
	45	27·413	2·359		45	38·846	3·807
	50	25·116	2·302		50	35·853	3·691
	55	22·664	2·237		55	32·353	3·535
	60	20·022	2·170		60	28·581	3·378
	65	17·191	2·102		65	24·540	3·224
	70	14·179	2·034		70	20·239	3·075
40	10	40·763	2·956	55	10	53·170	4·810
	15	40·023	2·974		15	52·454	4·834
	20	39·164	2·991		20	51·668	4·867
	25	37·969	2·954		25	50·596	4·826
	30	36·560	2·909		30	49·329	4·776
	35	34·888	2·857		35	47·829	4·721
	40	32·868	2·794		40	46·034	4·664
	45	30·501	2·715		45	43·800	4·583
	50	27·946	2·639		50	40·993	4·479
	55	25·218	2·555		55	37·357	4·303
	60	22·278	2·468		60	33·002	4·080
	65	19·128	2·382		65	28·336	3·863
	70	15·776	2·296		70	23·370	3·656

TABLE LIII Continued.

Age of A	Age of B	Single Premium	Annual Premium	Age of A	Age of B	Single Premium	Annual Premium
60	10	58·087	5·836	65	10	63·510	7·285
	15	57·403	5·863		15	62·870	7·313
	20	56·669	5·905		20	62·203	7·367
	25	55·675	5·803		25	61·311	7·325
	30	54·499	5·811		30	60·251	7·271
	35	53·103	5·755		35	58·990	7·213
	40	51·437	5·699		40	57·484	7·159
	45	49·367	5·622		45	55·620	7·085
	50	46·777	5·529		50	53·293	7·002
	55	43·439	5·371		55	50·302	6·858
	60	38·923	5·117		60	46·279	6·640
	65	33·419	4·795		65	40·576	6·270
	70	27·563	4·490		70	33·466	5·787

TABLE LIV.

Showing the value of an Assurance of £100 on a Single Life, according to the probabilities of living as observed by *M. De Parcieux.*

Age	4 per Cent	5 per Cent	Age	4 per Cent	5 per Cent	Age	4 per Cent	5 per Cent
6	23·550	18·843	36	34·915	28·300	66	66·573	60·838
7	23·258	18·471	37	35·558	28·886	67	67·885	62·295
8	23·092	18·233	38	36·323	29·605	68	69·158	63·714
9	22·992	18·052	39	37·119	30·357	69	70·388	65·090
10	23·046	18·033	40	37·950	31·148	70	71·562	66·405
11	23·273	18·195	41	38·819	31·981	71	72·754	67·748
12	23·677	18·543	42	39·719	32·857	72	73·869	69·024
13	24·100	18·910	43	40·665	33·781	73	74·977	70·257
14	24·538	19·290	44	41·650	34·743	74	76·069	71·500
15	24·992	19·690	45	42·677	35·776	75	77·135	72·714
16	25·462	20·110	46	43·750	36·852	76	78·262	74·010
17	25·481	20·452	47	44·781	37·886	77	79·346	75·262
18	26·281	20·814	48	45·862	38·976	78	80·365	76·438
19	26·715	21·186	49	46·896	40·024	79	81·408	77·648
20	27·162	21·576	50	47·977	41·129	80	82·323	78·710
21	27·534	21·890	51	49·019	42·190	81	83·192	79·724
22	27·923	22·210	52	50·019	43·205	82	83·985	80·643
23	28·319	22·548	53	51·058	44·271	83	84·850	81·652
24	28·731	22·890	54	52·142	45·390	84	85·846	82·838
25	29·154	23·252	55	53·181	46·467	85	86·830	83·995
26	29·592	23·624	56	54·265	47·590	86	87·750	85·086
27	30·046	24·010	57	55·396	48·776	87	88·550	86·033
28	30·515	24·414	58	56·484	49·919	88	89·573	87·262
29	31·000	24·833	59	57·615	51·121	89	90·592	88·486
30	31·500	25·271	60	58·796	52·367	90	91·588	89·686
31	32·019	25·724	61	60·027	53·686	91	92·535	90·838
32	32·558	26·195	62	61·308	55·071	92	93·415	91·890
33	33·115	26·690	63	62·562	56·429	93	94·304	92·971
34	33·692	27·205	64	63·869	57·857			
35	34·292	27·738	65	65·235	59·357			

TABLE LV.

Showing the value of an Assurance of £100 on Two Joint Lives, according to the probabilities of living as observed by *M. De Parcieux.*

Ages		3½ per Cent	4½ per Cent	Ages		3½ per Cent	4½ per Cent
10	10	37·784	30·937	30	60	65·446	58·828
	15	38·963	32·635		70	75·707	70·524
	20	40·565	33·344				
	30	43·243	35·915	35	35	49·246	41·711
	40	47·977	40·389		40	51·183	43·636
	50	55·876	48·523		45	54·168	46·667
	60	64·665	57·941		55	61·388	54·401
	70	75·378	70·067		65	70·654	64·658
15	15	40·040	33·051	40	40	52·799	45·268
	20	41·325	34·270		45	55·383	47·972
	25	42·499	35·321		50	58·643	51·452
	35	45·880	38·417		60	66·023	59·448
	45	52·123	44·622		70	75·899	70·070
	55	60·384	53·308				
	65	70·201	64·138	45	45	57·547	50·267
					50	60·374	53·316
20	20	42·475	35·389		55	63·964	56·580
	25	43·523	36·337		65	71·554	65·671
	30	44·853	37·577				
	40	49·181	41·672	50	50	62·714	55·878
	50	56·712	49·440		55	65·257	58·652
	60	65·213	58·570		60	68·409	62·135
	70	75·602	70·403		70	76·981	71·936
25	25	44·423	37·168	55	55	67·303	60·912
	30	45·027	38·279		60	69·941	63·854
	35	47·243	39·790		65	73·503	68·013
	45	52·088	45·548				
	55	60·918	53·915	60	60	71·990	66·166
	65	70·522	64·512		65	75·033	69·663
					70	78·571	72·341
30	30	46·651	39·240				
	35	48·065	40·578	65	65	77·246	73·784
	40	50·256	42·736		70	80·190	75·683
	50	57·239	50·000				

TABLE LVI.

Showing the value of an Assurance of £100 on a Single Life, according to the probabilities of living as observed in *Sweden*.

Age	Males		Females		Age	Males		Females	
	4 per Cent	5 per Cent	4 per Cent	5 per Cent		4 per Cent	5 per Cent	4 per Cent	5 per C
6	24·531	19·519	22·904	18·081	36	39·996	33·305	37·392	30·75
7	24·258	19·157	22·573	17·662	37	40·838	34·129	38·192	31·51
8	24·134	18·948	22·454	17·452	38	41·715	34·990	39·023	32·31
9	24·173	18·935	22·406	17·414	39	42·631	35·895	39·888	33·10
10	24·327	18·981	22·658	17·500	40	43·585	36·852	40·765	34·01
11	24·615	19·190	22·919	17·686	41	44·515	37·786	41·596	34·82
12	25·034	19·543	23·261	17·957	42	45·400	38·667	42·331	35·53
13	25·469	19·910	23·692	18·310	43	46·215	39·476	43·077	36·25
14	25·977	20·362	24·204	18·707	44	47·065	40·324	43·861	37·00
15	26·519	20·838	24·738	19·238	45	47·932	41·265	44·681	37·80
16	27·084	21·348	25·292	19·733	46	48·858	42·129	45·573	38·69
17	27·681	21·886	25·808	20·180	47	49·804	43·060	46·561	39·67
18	28·296	22·452	26·342	20·657	48	50·788	44·105	47·615	40·74
19	28·877	22·976	26·873	21·129	49	51·815	45·161	48·719	41·88
20	29·481	23·529	27·415	21·618	50	52·819	46·200	49·812	43·00
21	30·031	24·024	27·981	22·114	51	53·731	47·143	50·888	44·11
22	30·608	24·550	28·565	22·643	52	54·673	48·119	51·954	45·20
23	31·204	25·086	29·177	23·195	53	55·650	49·133	53·0[illegible]6	46·28
24	31·762	25·586	29·800	23·707	54	56·658	50·190	54·0[illegible]8	47·41
25	32·338	26·110	30·434	24·352	55	57·700	51·290	55·223	48·61
26	32·938	26·657	31·096	24·967	56	58·781	52·438	56·493	49·86
27	33·562	27·229	31·727	25·543	57	59·904	53·638	57·646	51·17
28	34·211	27·829	32·354	26·119	58	61·000	54·814	58·877	52·49
29	34·885	28·457	32·973	26·680	59	62·134	56·038	60·102	53·88
30	35·573	29·100	33·611	27·270	60	63·308	57·319	61·388	55·21
31	36·250	29·733	34·215	27·829	61	64·458	58·571	62·542	56·45
32	36·942	30·386	34·812	28·405	62	65·577	59·800	63·642	57·64
33	37·602	31·062	35·435	28·938	63	66·658	60·980	64·746	58·84
34	38·408	31·776	36·042	29·465	64	67·702	62·200	65·885	60·08
35	39·185	32·524	36·673	30·076	65	68·885	63·448	67·054	61·37

TABLE LVI Continued.

Age	Males		Females	
	4 per Cent	5 per Cent	4 per Cent	5 per Cent
66	70·031	64·724	68·262	62·710
67	71·196	66·029	69·500	64·090
68	72·304	67·276	70·785	65·524
69	73·335	68·438	72·104	67·019
70	74·538	69·576	73·473	68·576
71	75·315	70·676	74·754	70·033
72	76·231	71·714	75·919	71·367
73	77·154	72·767	76·931	72·524
74	77·985	73·710	77·723	73·424
75	78·896	74·752	78·531	74·343
76	79·796	75·790	79·358	75·290
77	80·077	76·805	80·212	76·271
78	81·662	77·948	81·104	77·300
79	82·646	79·095	82·046	78·400
80	83·615	80·229	83·069	79·596
81	84·550	81·329	84·058	80·757
82	85·415	82·352	84·981	81·848
83	86·154	83·224	85·773	82·786
84	86·642	83·795	86·311	83·429
85	87·034	84·257	86·338	83·448
86	87·380	84·657	86·469	83·590
87	87·869	85·224	86·804	83·976
88	88·635	86·133	87·331	84·590
89	89·623	87·314	88·046	85·433
90	90·704	88·610	88·950	86·510
91	91·715	89·829	89·892	87·638
92	92·942	91·314	90·965	88·929
93	94·319	92·996	92·034	90·219
94	95·231	94·105	93·081	91·486
95			94·062	92·681
96			94·923	93·729

TABLE LVII.

Showing the value of an Assurance of £100 on Two Joint Lives, according to the probabilities o living as observed in *Sweden*.

Ages		4 per Cent	Ages		4 per Cent	Ages		4 per Cent
10	10	34·073	30	54	59·950	55	55	67·327
	16	35·658		60	65·512		61	70·942
	22	37·888		66	71·227		67	75·100
	28	40·446		72	77·381		73	79·881
	34	43·304	35	35	49·581		79	84·038
	40	47·142		41	52·458		85	87·509
	46	51·177		47	55·556		91	92·008
	52	56·310		53	59·800	60	60	72·577
15	15	36·404		59	64·946		66	75·965
	21	38·458		65	70·465		72	80·195
	27	40·800		71	76·565		78	83·898
	33	43·402		77	81·412		84	87·500
	39	46·923	40	40	53·985		90	91·027
	45	50·819		46	56·592	65	65	77·381
	51	55·715		52	60·354		71	80·838
	57	61·204		58	64·968		77	83·923
20	20	39·684		64	70·142		83	87·492
	26	41·754		70	75·946		89	90·058
	32	44·185		76	80·865		95	94·400
	38	47·231		82	85·731	70	70	82·334
	44	50·958	45	45	57·869		76	84·823
	50	55·512		51	61·042		82	87·865
	56	60·662		57	64·996		88	89·765
	62	66·700		63	69·862		94	94·177
25	25	42·888		69	75·285	75	75	85·909
	31	45·062		75	80·277		81	88·396
	37	47·696		81	85·177		87	89·981
	43	51·219		87	88·142		93	93·942
	49	55·338	50	50	62·665	80	80	89·392
	55	60·204		56	65·869		86	90·708
	61	66·065		62	70·196		92	93·700
	67	72·019		68	75·050	85	85	91·492
30	30	46·288		74	79·981		91	93·365
	36	48·500		80	84·654	90	90	93·392
	42	51·758		86	87·804	95	95	96·062
	48	55·362		92	93·008			

TABLE LVIII.

Showing the present value of £1 to be received at the end of any number of years.

Years	4 per Cent	4½ per Cent	5 per Cent	Years	4 per Cent	4½ per Cent	5 per Cent
1	·96154	·95694	·95238	31	·29646	·25550	·22036
2	·92456	·91573	·90703	32	·28506	·24450	·20987
3	·88900	·87630	·86384	33	·27409	·23397	·19987
4	·85480	·83856	·82270	34	·26355	·22390	·19035
5	·82193	·80245	·78353	35	·25342	·21425	·18129
6	·79031	·76790	·74622	36	·24367	20503	·17266
7	·75992	·73483	·71068	37	·23430	·19620	·16444
8	·73069	·70319	·67684	38	·22529	·18775	·15661
9	·70259	·67290	·64461	39	·21662	17967	·14915
10	·67556	·64393	·61391	40	·20829	·17193	·14205
11	·64958	·61620	·58468	41	·20028	·16453	·13528
12	·62460	·58966	·55684	42	·19257	·15744	·12884
13	·60057	·56427	·53032	43	·18517	·15066	·12270
14	·57748	·53997	·50507	44	·17805	·14417	·11686
15	·55526	·51672	·48102	45	·17120	·13796	·11130
16	·53391	·49447	·45811	46	·16461	·13202	·10600
17	·51337	·47318	·43630	47	·15828	·12634	·10095
18	·49363	·45280	·41552	48	·15219	·12090	·09614
19	·47464	·43330	·39573	49	·14634	·11569	·09156
20	·45639	·41464	·37689	50	·14071	·11071	·08720
21	·43883	·39679	·35894	51	·13530	·10594	·08305
22	·42196	·37970	·34185	52	·13010	·10138	·07910
23	·40573	·36335	·32557	53	·12509	·09701	·07533
24	·39012	·34770	·31007	54	·12028	·09284	·07174
25	·37512	·33273	·29530	55	·11566	·08884	·06833
26	·36069	·31840	·28124	56	·11121	·08501	06507
27	·34682	·30469	·26785	57	·10693	·08135	·06197
28	·33348	·29157	·25509	58	·10282	·07785	·05902
29	·32065	·27902	·24295	59	·09886	·07450	·05621
30	·30832	·26700	·23138	60	·09506	·07129	·05354

TABLE LVIII Continued.

Years	4 per Cent	4½ per Cent	5 per Cent	Years	4 per Cent	4½ per Cent	5 per Cent
61	·09140	·06822	·05099	81	·04172	·02820	·01922
62	·08789	·06528	·04856	82	04011	·02707	·01830
63	·08451	·06247	·04625	83	·03857	·02590	·01743
64	·08126	·05978	·04404	84	·03709	·02479	·01660
65	·07813	·05721	·04195	85	·03566	·02372	·01581
66	·07513	·05474	·03995	86	·03429	·02270	·01506
67	·07224	·05239	·03805	87	·03297	·02172	·01434
68	·06946	·05013	·03623	88	·03170	·02079	·01366
69	·06679	·04797	·03451	89	·03048	·01989	·01301
70	·06422	·04590	·03287	90	·02931	01903	01239
71	·06175	·04393	·03130	91	·02818	·01821	·01180
72	·05937	·04204	·02981	92	·02710	·01743	·01124
73	·05709	·04023	·02839	93	·02606	·01668	·01070
74	·05490	·03849	·02704	94	·02505	·01596	·01019
75	·05278	·03684	·02575	95	·02409	·01527	·00971
76	·05075	·03525	·02453	96	·02316	·01462	·00924
77	·04880	·03373	·02336	97	·02227	·01399	·00880
78	·04692	·03228	·02225	98	·02142	·01338	·00838
79	·04512	·03089	·02119	99	·02059	·01281	·00798
80	·04338	·02956	·02018	100	·01980	·01226	·00760

TABLE LIX.

Showing the present value of £1 *per annum* for any number of years.

Years	4 per Cent	4½ per Cent	5 per Cent	Years	4 per Cent	4½ per Cent	5 per Cent
1	·962	·957	·952	31	17·588	16·544	15·593
2	1·886	1·873	1·859	32	17·874	16·789	15·803
3	2·775	2·749	2·723	33	18·148	17·023	16·003
4	3·630	3·588	3·546	34	18·411	17·247	16·193
5	4·452	4·390	4·329	35	18·665	17·461	16·374
6	5·242	5·158	5·076	36	18·908	17·666	16·547
7	6·002	5·893	5·786	37	19·143	17·862	16·711
8	6·733	6·596	6·463	38	19·368	18·050	16·868
9	7·435	7·269	7·108	39	19·584	18·230	17·017
10	8·111	7·913	7·722	40	19·793	18·402	17·159
11	8·760	8·529	8·306	41	19·993	18·566	17·294
12	9·385	9·119	8·863	42	20·186	18·724	17·423
13	9·986	9·683	9·394	43	20·371	18·874	17·546
14	10·563	10·223	9·899	44	20·549	19·018	17·663
15	11·118	10·740	10·380	45	20·720	19·156	17·774
16	11·652	11·234	10·838	46	20·885	19·288	17·880
17	12·166	11·707	11·274	47	21·043	19·415	17·981
18	12·659	12·160	11·690	48	21·195	19·536	18·077
19	13·134	12·593	12·085	49	21·341	19·651	18·169
20	13·590	13·008	12·462	50	21·482	19·762	18·256
21	14·029	13·405	12·821	51	21·617	19·868	18·339
22	14·451	13·784	13·163	52	21·748	19·969	18·418
23	14·857	14·148	13·489	53	21·873	20·066	18·493
24	15·247	14·495	13·799	54	21·993	20·159	18·565
25	15·622	14·828	14·094	55	22·109	20·248	18·633
26	15·983	15·147	14·375	56	22·220	20·333	18·699
27	16·330	15·451	14·643	57	22·327	20·414	18·761
28	16·663	15·713	14·898	58	22·430	20·492	18·820
29	16·984	16·022	15·141	59	22·528	20·567	18·876
30	17·292	16·289	15·372	60	22·623	20·638	18·929

TABLE LIX Continued.

Years	4 per Cent	4½ per Cent	5 per Cent	Years	4 per Cent	4½ per Cent	5 per Cent
61	22·715	20·706	18·980	86	24·143	21·718	19·699
62	22·803	20·772	19·029	87	24·176	21·740	19·713
63	22·887	20·834	19·075	88	24·207	21·760	19·727
64	22·969	20·894	19·119	89	24·238	21·780	19·740
65	23·047	20·951	19·161	90	24·267	21·799	19·752
66	23·122	21·006	19·201	91	24·295	21·817	19·764
67	23·194	21·058	19·239	92	24·323	21·835	19·775
68	23·264	21·108	19·275	93	24·349	21·852	19·786
69	23·330	21·156	19·310	94	24·374	21·868	19·796
70	23·395	21·202	19·343	95	24·398	21·883	19·806
71	23·456	21·246	19·374	96	24·421	21·897	19·815
72	23·516	21·288	19·404	97	24·443	21·911	19·824
73	23·573	21·328	19·432	98	24·465	21·925	19·832
74	23·628	21·367	19·459	99	24·485	21·938	19·840
75	23·680	21·404	19·485	100	24·505	21·950	19·848
76	23·731	21·439	19·509	Perp	25·000	22·222	20·000
77	23·780	21·473	19·533				
78	23·827	21·505	19·555				
79	23·872	21·536	19·576				
80	23·915	21·565	19·596				
81	23·957	21·594	19·616				
82	23·997	21·621	19·634				
83	24·036	21·647	19·651				
84	24·073	21·671	19·668				
85	24·109	21·695	19·684				

THE END.

ERRATA.

Page	*line*	*for*	*read*
6	last,	§ 2	§ 27
41	17,	last	twelfth
131	11,	$(1-\phi-\chi)\times(A)^{\mathrm{d}}-(1-\chi)\times(AB)^{\mathrm{d}}$	$(1-\varphi)\times(A)^{\mathrm{d}}-(AB)^{\mathrm{d}}$
144	last,	§ 197	§ 200.
153	19,	that sum	the nº. of years purchase
168	10,	$[(A+B(1-\varrho A)]$	$[A+B(1-\varrho A)]$
175	11,	$\frac{f}{\overset{2}{A}}$	$\frac{f}{\overset{1}{A}}$
240	last,	$\frac{1+ABC}{(1+\varrho)}$	$\frac{1-\varrho ABC}{(1+\varrho)}$
278	13,	$(1+\varrho)^{-n+1}$	$(1+\varrho)^{-(n+1)}$
329	5,	*AA*	AA
345	19,	·05	·5
352	18,	formula	formulæ
355	17,	§ 23	§ 23 and 24
380	14,	first	second
431	7,	22·234	24·107
449	7 from bottom	*dele*	by Table LVIII
514	in the first Table,	22·78 *ought to be*	22·28

In my *Doctrine of Interest and Annuities*, page 134, line 5, I have stated that the formula there deduced " is the very formula given by Dr. Halley." But this is not exactly the case, since Dr. Halley makes $\gamma=\frac{6(2\nu+n+1)+6}{n^2}$: and I am not aware how the difference arises. Both the formulæ, however, give the value sufficiently near for general use.